Solutions

Final Exam (Fall 2013)

PHYS 320: Electricity and Magnetism I

Date: 2013 Dec 13

1. **(20 points.)** The electric potential due to an infinitely thin plate (or a large disc of radius R on the xy-plane with $|x|, |y|, |z| \ll R$) with uniform charge density σ is given by the expression

$$\phi(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0}\big[R - |z|\big]. \tag{1}$$

Find the (simplified) expression for the electric field due to the plane by evaluating the gradient of the above electric potential,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{2}$$

2. **(20 points.)** Consider a solid sphere of radius R with total charge Q distributed inside the sphere with a charge density

$$\rho(\mathbf{r}) = br^3\,\theta(R - r), \tag{3}$$

where r is the distance from the center of sphere, and $\theta(x) = 1$, if $x > 0$, and 0 otherwise.

   (a) Integrating the charge density over all space gives you the total charge Q. Thus, determine the constant b in terms of Q and R.

   (b) Using Gauss's law find the electric field inside and outside the sphere.

   (c) Plot the electric field as a function of r.

3. **(20 points.)** Consider a semi-infinite dielectric slab described by

$$\varepsilon(z) = \begin{cases} \varepsilon_2 & z < a, \\ \varepsilon_1 < \varepsilon_2 & a < z. \end{cases} \tag{4}$$

A positive point charge q is embedded at position $\mathbf{r}'$ (with $a < z'$) on one side of the interface. The electric field for this configuration is given by

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_1}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{q_{\text{im}}}{4\pi\varepsilon_1}\frac{\mathbf{r} - \mathbf{r}'_{\text{im}}}{|\mathbf{r} - \mathbf{r}'_{\text{im}}|^3}, \tag{5}$$

where the image charge is

$$q_{\text{im}} = -q\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2}, \tag{6}$$

and the position of the image charge is

$$\mathbf{r}'_{\text{im}} = \mathbf{r}' - 2(z' - a)\hat{\mathbf{z}}. \tag{7}$$

Prob. 1, Final Exam

$$\phi(\vec{r}) = \frac{\sigma}{2\epsilon_0}\left[R - |z|\right]$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}\left(\frac{\sigma}{2\epsilon_0}\left[R - |z|\right]\right)$$

$$= -\hat{z}\frac{\partial}{\partial z}\left(\frac{\sigma}{2\epsilon_0}\left[R - |z|\right]\right)$$

$$= \frac{\sigma}{2\epsilon_0}\,\hat{z}\,\frac{\partial}{\partial z}|z|$$

$$= \begin{cases} \dfrac{\sigma}{2\epsilon_0}\,\hat{z} & z > 0 \\ -\dfrac{\sigma}{2\epsilon}\,\hat{z} & z < 0 \end{cases}$$

$$\frac{d}{dz}|z| = \begin{cases} +1, & z>0, \\ -1, & z<0, \end{cases}$$

Prob 2, Final Exam

(a)

$$Q = \int d^3r\ \rho(\vec{r})$$

$$= 4\pi \int_0^R r^2 dr\ b r^3$$

$$= 4\pi b\,\frac{R^6}{6}$$

$$b = \frac{3}{2\pi R^6}.$$

(b) $$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{en}}{\epsilon_0}$$

For $\underline{\underline{R < r}}$, $Q_{en} = Q$

$$E \, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}, \qquad R < r.$$

For $\underline{\underline{r < R}}$

$$Q_{en} = Q \, \frac{\rho(r) \, \cancel{\tfrac{4}{3}} \cancel{\pi} r^3}{\rho(R) \, \cancel{\tfrac{4}{3}} \cancel{\pi} R^3}$$

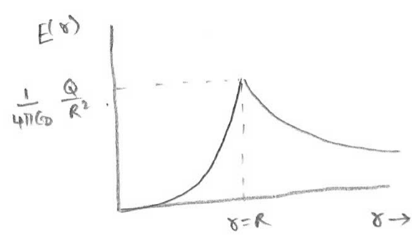
$$= Q \, \frac{b r^3 \, r^3}{b R^3 \, R^3} = Q \, \frac{r^6}{R^6}$$

$$E \, 4\pi r^2 = \frac{Q_{en}}{\epsilon_0} = \frac{Q}{\epsilon_0} \frac{r^6}{R^6}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \frac{r^4}{R^4}, \qquad r < R.$$

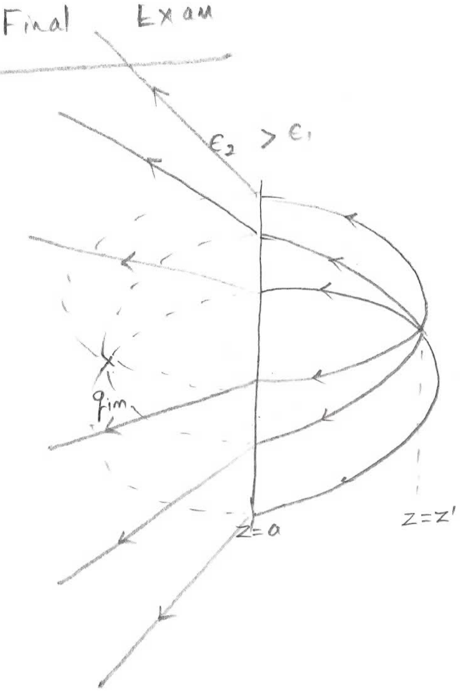
$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \dfrac{Q}{R^2} \dfrac{r^4}{R^4}, & r < R, \\[2ex] \dfrac{1}{4\pi\epsilon_0} \dfrac{Q}{r^2}, & R < r. \end{cases}$$

(c)

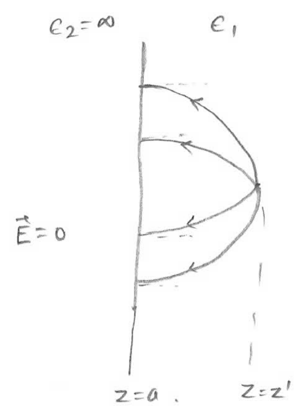


Prob 3, Final Exam

(a)



(b)



electric field lines are perpendicular to the plate at $z=a$.

Prob 4, Final Exam

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{it\cos\alpha}.$$

$$\operatorname{Im}\left[J_0(t)\right] = \frac{1}{2i} \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{it\cos\alpha} - \frac{1}{2i} \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{-it\cos\alpha}.$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi} \, \sin\left(it\cos\alpha\right)$$

$$= \int_0^{\pi} \frac{d\alpha}{2\pi} \, \sin\left(it\cos\alpha\right) + \int_{\pi}^{2\pi} \frac{d\alpha}{2\pi} \, \sin\left(it\cos\alpha\right)$$

$$\alpha' = \alpha - \pi$$

$$d\alpha' = d\alpha$$

$$= \int_0^{\pi} \frac{d\alpha}{2\pi} \, \sin\left(it\cos\alpha\right) + \int_0^{\pi} \frac{d\alpha'}{2\pi} \, \sin\left(it\cos(\pi+\alpha')\right)$$

$$= \int_0^{\pi} \frac{d\alpha}{2\pi} \, \sin\left(it\cos\alpha\right) - \int_0^{\pi} \frac{d\alpha'}{2\pi} \, \sin\left(it\cos\alpha'\right)$$

$$= 0.$$

Prob. 5, Final Exam

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha}$$

$$\left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} + 1\right] J_0(t)$$

$$= \left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} + 1\right] \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha}$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[(i\cos\alpha)^2 + \frac{1}{t}(i\cos\alpha) + 1\right] e^{it\cos\alpha}$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[\sin^2\alpha + \frac{i}{t}\cos\alpha\right] e^{it\cos\alpha}$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi} \frac{d}{d\alpha}\left[\frac{i}{t} \sin\alpha \, e^{it\cos\alpha}\right]$$

$$= \frac{1}{2\pi}\frac{i}{t} \sin\alpha \, e^{it\cos\alpha} \Big|_{\alpha=0}^{\alpha=2\pi}$$

$$= 0.$$