Final Exam (Fall 2013)

PHYS 320: Electricity and Magnetism I

Date: 2013 Dec 13

1. (**20 points.**) The electric potential due to an infinitely thin plate (or a large disc of radius R on the xy-plane with $|x|, |y|, |z| \ll R$) with uniform charge density σ is given by the expression
$$\phi(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0}\big[R - |z|\big]. \tag{1}$$
Find the (simplified) expression for the electric field due to the plane by evaluating the gradient of the above electric potential,
$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{2}$$

2. (**20 points.**) Consider a solid sphere of radius R with total charge Q distributed inside the sphere with a charge density
$$\rho(\mathbf{r}) = br^3\,\theta(R - r), \tag{3}$$
where r is the distance from the center of sphere, and $\theta(x) = 1$, if $x > 0$, and 0 otherwise.
   (a) Integrating the charge density over all space gives you the total charge Q. Thus, determine the constant b in terms of Q and R.
   (b) Using Gauss's law find the electric field inside and outside the sphere.
   (c) Plot the electric field as a function of r.

3. (**20 points.**) Consider a semi-infinite dielectric slab described by
$$\varepsilon(z) = \begin{cases} \varepsilon_2 & z < a, \\ \varepsilon_1 < \varepsilon_2 & a < z. \end{cases} \tag{4}$$
A positive point charge q is embedded at position $\mathbf{r}'$ (with $a < z'$) on one side of the interface. The electric field for this configuration is given by
$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_1}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{q_{\text{im}}}{4\pi\varepsilon_1}\frac{\mathbf{r} - \mathbf{r}'_{\text{im}}}{|\mathbf{r} - \mathbf{r}'_{\text{im}}|^3}, \tag{5}$$
where the image charge is
$$q_{\text{im}} = -q\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2}, \tag{6}$$
and the position of the image charge is
$$\mathbf{r}'_{\text{im}} = \mathbf{r}' - 2(z' - a)\hat{\mathbf{z}}. \tag{7}$$

(a) Draw the electric field lines for this configuration ($\varepsilon_2 > \varepsilon_1$).

(b) Draw the electric field lines for the case $\varepsilon_2 \to \infty$.

4. (**20 points.**) Bessel function of zeroth order is defined by the integral representation

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{it\cos\alpha}. \tag{8}$$

Verify that $J_0(t)$ is indeed a real function by showing that

$$\text{Im}\,[J_0(t)] = 0. \tag{9}$$

5. (**20 points.**) Bessel function of zeroth order is defined by the integral representation

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{it\cos\alpha}. \tag{10}$$

Verify, by substitution, that it satisifes the differential equation

$$\left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} + 1\right] J_0(t) = 0. \tag{11}$$