SOLUTIONS

Homework No. 02 (Fall 2013)

PHYS 320: Electricity and Magnetism I

Due date: Friday, 2013 Sep 13, 4.30pm

1. Laplacian:
   (a) Problem 1.26 in Griffiths 4th edition.
2. Integral calculus:
   (a) Problems 1.32 in Griffiths 4th edition.
   (b) Problems 1.33 in Griffiths 4th edition.
   (c) Problems 1.34 in Griffiths 4th edition.
3. Curvilinear coordinates:
   (a) Problems 1.38 in Griffiths 4th edition. (Hint: Use HW-01, prob 5)
   (b) Problems 1.39 in Griffiths 4th edition.
   (c) Problems 1.40 in Griffiths 4th edition.
   (d) Problems 1.62 (only part a) in Griffiths 4th edition.
   (e) Show that
$$\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}} = -\big[\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\big], \tag{1}$$
   where (r, θ, ϕ) are spherical coordinates and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are the respective unit vectors in spherical coordinates. Sketch $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, and $\partial\hat{\boldsymbol{\phi}}/\partial\phi$ to illustrate their relative directions.
4. Delta function:
   (a) Problems 1.44 in Griffiths 4th edition.
   (b) Problems 1.45 in Griffiths 4th edition.
   (c) Problems 1.47 in Griffiths 4th edition.
   (d) Problems 1.48 in Griffiths 4th edition.

Solutions to HW-02 (Fall 2013) - PHYS-320

Prob 1 : Griffiths 1.26

(a) $T_a = x^2 + 2xy + 3z + 4$

$$\vec{\nabla} T_a = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] T_a$$

$$= \hat{i}(2x+2y) + \hat{j}\,2x + \hat{k}\,3$$

$$\vec{\nabla}\cdot\vec{\nabla} T_a = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\hat{i}(2x+2y) + \hat{j}\,2x + \hat{k}\,3\right]$$

$$= \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial z}(3)$$

$$= 2 + 0 + 0 = 2$$

(b) $T_b = \sin x \sin y \sin z$

$$\vec{\nabla} T_b = \hat{i}\cos x \sin y \sin z + \hat{j}\sin x \cos y \sin z + \hat{k}\sin x \sin y \cos z$$

$$\nabla^2 T_b = -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$$

$$= -3T_b$$

(c) $T_c = e^{-5x}\sin 4y \cos 3z$

$$\vec{\nabla} T_c = \hat{i}(-5e^{-5x}\sin 4y\cos 3z) + \hat{j}\,4e^{-5x}\cos 4y\cos 3z - \hat{k}\,3e^{-5x}\sin 4y \sin 3z$$

$$\nabla^2 T_c = 25T_c - 16T_c - 9T_c$$

$$= 0$$

(d) $\vec{v} = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$

$$\nabla^2\vec{v} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\vec{v}$$

$$= (2\hat{i} + 0\hat{j} + 0\hat{k}) + (0\hat{i} + 0\hat{j} + 0\hat{k}) + (0\hat{i} + 6x\hat{j} + 0\hat{k})$$

$$= 2\hat{i} + 6x\hat{j}$$

Prob 2 (a) : Griffiths 1.32

$$\int_{\vec a}^{\vec b} d\vec l \cdot \vec\nabla T = T(\vec b) - T(\vec a)$$

$\vec a = (0,0,0)$
$\vec b = (1,1,1)$

$T = x^2 + 4xy + 2yz^3$

$\vec\nabla T = \hat i\,(2x+4y) + \hat j\,(4x+2z^3) + \hat k\, 6yz^2$

$d\vec l \cdot \vec\nabla T = dx\,(2x+4y) + dy\,(4x+2z^3) + dz\,(6yz^2)$

R.H.S $= T(\vec b) - T(\vec a)$

$= 7 - 0 = 7$

L.H.S (path (a)) $= \underset{dy=0,\ y=0,\ z=0,\ dz=0}{\int_0^1 dx\,(2x+4y)} + \underset{dx=0,\ dz=0,\ x=1,\ z=0}{\int_0^1 dy\,(4x+2z^3)} + \underset{dx=0,\ dy=0,\ x=1,\ y=1}{\int_0^1 dz\,(6yz^2)}$

$= 1 + 4 + 2 = 7$ ✓

L.H.S. (path (b)) $= \underset{dx=0\ dy=0,\ x=0,\ y=0}{\int_0^1 dz\,(6yz^2)} + \underset{dx=0,\ dz=0,\ x=0,\ z=1}{\int_0^1 dy\,(4x+2z^3)} + \underset{dy=0,\ dz=0,\ y=1,\ z=1}{\int_0^1 dx\,(2x+4y)}$

$= 0 + 2 + 5 = 7$ ✓

L.H.S. (path (c)) : $dz = 2x\,dx$ $\quad z = x^2$
$dy = dx$ $\quad y = x$

$d\vec l \cdot \vec\nabla T = dx\,(2x+4x) + dx\,(4x+2x^6) + 2x\,dx\,(6x\,x^4)$

$= dx\,(6x + 4x + 2x^6 + 12x^6)$

$= dx\,(10x + 14x^6)$

$\int_{\vec a}^{\vec b} d\vec l \cdot \vec\nabla T = \int_0^1 dx\,(10x + 14x^6)$

$= 5 + 2 = 7$ ✓

Prob 2 (b) : Griffiths 1.33

V: cube $(0,0,0) - (2,2,2)$

$$\int_V d^3x \ \vec{\nabla}\cdot\vec{v} = \oint_S d\vec{a}\cdot\vec{v}$$

$$\vec{v} = xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{v} = y + 2z + 3x.$$

$$\text{L.H.S.} = \int_0^2 dx \int_0^2 dy \int_0^2 dz \ (y + 2z + 3x)$$

$$= \int_0^2 dx \int_0^2 dy \ (yz + z^2 + 3xz)\Big|_0^2 = \int_0^2 dx \int_0^2 dy \ (2y + 4 + 6x)$$

$$= \int_0^2 dx \ (y^2 + 4y + 6xy)\Big|_0^2 = \int_0^2 dx \ (4 + 8 + 12x)$$

$$= (12x + 6x^2)\Big|_0^2 = 24 + 24 = 48$$

$$\text{R.H.S.} = \int_0^2 dy \int_0^2 dz \left[+\hat{i}\cdot(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k})_{x=2} - \hat{i}\cdot(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k})_{x=0} \right]$$

$$+ \int_0^2 dx \int_0^2 dz \left[+\hat{j}\cdot(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k})_{y=2} - \hat{j}\cdot(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k})_{y=0} \right]$$

$$+ \int_0^2 dx \int_0^2 dy \left[+\hat{k}\cdot(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k})_{z=2} - \hat{k}\cdot(xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k})_{z=0} \right]$$

$$= \int_0^2 dy \int_0^2 dz \ 2y + \int_0^2 dx \int_0^2 dz \ 4z + \int_0^2 dx \int_0^2 dy \ 6x$$

$$= \int_0^2 dy \ 4y + \int_0^2 dz \ 8z + \int_0^2 dx \ 12x$$

$$= 8 + 16 + 24 = 48 \ \checkmark$$

Prob. 2 (c) : Griffiths 1.34

$$\int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{v}) = \oint d\vec{l} \cdot \vec{v}$$

$$\vec{v} = xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}$$

$$\vec{\nabla} \times \vec{v} = \hat{i}\,(0-2y) - \hat{j}\,(3z-0) + \hat{k}\,(0-x)$$

$$= -2y\,\hat{i} - 3z\,\hat{j} - x\,\hat{k}$$

$$\int_S d\vec{a} = \int_0^2 dy \int_0^{2-y} dz\ \hat{i} \quad \text{with } x=0.$$

$$\text{L.H.S.} = \int_S d\vec{a} \cdot \vec{\nabla} \times \vec{v} = \int_0^2 dy \int_0^{2-y} dz\,(-2y)$$

$$= \int_0^2 dy\,(-2y)(2-y)$$

$$= \left(-4\frac{y^2}{2} + 2\frac{y^3}{3}\right)\Bigg|_0^2 = -8 + \frac{16}{3} = -\frac{8}{3}$$

$$d\vec{l} \cdot \vec{v} = dx\,(xy) + dy\,(2yz) + dz\,(3zx)$$

$$\text{R.H.S.} = \underset{\substack{dx=0,\ dy=0 \\ x=0,\ y=0}}{\int_{-2}^{0} dz\,(3zx)} + \underset{\substack{dx=0,\ dz=0, \\ x=0,\ z=0}}{\int_0^2 dy\,(2yz)} + \underset{\substack{z=2-y,\ dz=-dy,\ dx=0 \\ x=0}}{\int_2^0 \left[dy\,\{2y(2-y)\} + (-dy)(3(2-y)\,0)\right]}$$

$$= 0 + 0 + \int_2^0 dy\ 2y(2-y) = -\frac{8}{3} \quad \checkmark$$

Prob (3) (a) : Griffiths 1.38

$$x = r \sin\theta \cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r \sin\theta \sin\phi \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

$$z = r \cos\theta \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\hat{r} = \frac{\vec{\nabla} r}{|\vec{\nabla} r|} = \frac{1}{|\vec{\nabla} r|}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$$

$$= \sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\ \hat{k}$$

$$\hat{\theta} = \frac{\vec{\nabla}\theta}{|\vec{\nabla}\theta|} = \frac{1}{|\vec{\nabla}\theta|}\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad \frac{\partial}{\partial\theta}\tan\theta = \sec^2\theta$$

$$= \frac{\cos^2\theta}{|\vec{\nabla}\theta|}\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\frac{\sqrt{x^2+y^2}}{z}$$

$$= \frac{\cos^2\theta}{|\vec{\nabla}\theta|}\left[\frac{x}{z\,r\sin\theta}\hat{i} + \frac{y}{z\,r\sin\theta}\hat{j} - \hat{k}\frac{r\sin\theta}{z^2}\right]$$

$$= \frac{1}{|\vec{\nabla}\theta|}\left[\frac{1}{r}\cos\theta\cos\phi\ \hat{i} + \frac{1}{r}\cos\theta\sin\phi\ \hat{j} - \hat{k}\sin\theta\right]$$

$$= \cos\theta\cos\phi\ \hat{i} + \cos\theta\sin\phi\ \hat{j} - \sin\theta\ \hat{k}$$

$$\hat{\phi} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{1}{|\vec{\nabla}\phi|}\cos^2\phi\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\frac{y}{x}$$

$$= \frac{1}{|\vec{\nabla}\phi|}\cos^2\phi\left[-\frac{y}{x^2}\hat{i} + \frac{1}{x}\hat{j} + 0\ \hat{k}\right]$$

$$= \frac{1}{|\vec{\nabla}\phi|}\left[-\frac{\sin\phi}{r\sin\theta}\hat{i} + \frac{\cos\phi}{r\sin\theta}\hat{j} + 0\ \hat{k}\right]$$

$$= -\sin\phi\ \hat{i} + \cos\phi\ \hat{j} + 0\ \hat{k}$$

Thus,

$$\hat{r} = \sin\theta \cos\phi \ \hat{i} + \sin\theta \sin\phi \ \hat{j} + \cos\theta \ \hat{k}$$

$$\hat{\theta} = \cos\theta \cos\phi \ \hat{i} + \cos\theta \sin\phi \ \hat{j} - \sin\theta \ \hat{k}$$

$$\hat{\phi} = -\sin\phi \ \hat{i} + \cos\phi \ \hat{j} + 0 \ \hat{k}.$$

$$\hat{r} \cdot \hat{r} = 1 \qquad \hat{\theta} \cdot \hat{r} = 0 \qquad \hat{\phi} \cdot \hat{r} = 0$$

$$\hat{r} \cdot \hat{\theta} = 0 \qquad \hat{\theta} \cdot \hat{\theta} = 1 \qquad \hat{\phi} \cdot \hat{\theta} = 0$$

$$\hat{r} \cdot \hat{\phi} = 0 \qquad \hat{\theta} \cdot \hat{\phi} = 0 \qquad \hat{\phi} \cdot \hat{\phi} = 1$$

$$\hat{r} \times \hat{\theta} = \hat{i}\left(-\sin^2\theta \sin\phi - \cos^2\theta \sin\phi\right) - \hat{j}\left(-\sin^2\theta \cos\phi - \cos^2\theta \cos\phi\right)$$
$$+ \hat{k}\left(\sin\theta\cos\theta \sin\phi \cos\phi - \sin\theta\cos\theta \sin\phi \cos\phi\right)$$
$$= -\sin\phi \ \hat{i} + \cos\phi \ \hat{j} + 0 \ \hat{k} = \hat{\phi}.$$

$$\hat{\theta} \times \hat{\phi} = \hat{i}\left(0 + \sin\theta\cos\phi\right) - \hat{j}\left(0 - \sin\theta \sin\phi\right) + \hat{k}\left(\cos\theta\right)$$
$$= \hat{r}$$

$$\hat{\phi} \times \hat{r} = \hat{i}\left(\cos\phi\cos\theta - 0\right) - \hat{j}\left(-\sin\phi\cos\theta - 0\right) + \hat{k}\left(-\sin^2\phi \sin\theta - \sin\theta\cos^2\phi\right)$$
$$= \cos\theta\cos\phi \ \hat{i} + \cos\theta \sin\phi \ \hat{j} - \sin\theta \ \hat{k}$$
$$= \hat{\theta}$$

We can write

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \underbrace{\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}}_{R} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

$$\begin{pmatrix} \hat{i} \\ \hat{j} \\ k \end{pmatrix} = R^{-1} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

$$\det R = -\sin\phi\,(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi) - \cos\phi\,(-\sin^2\theta\cos\phi - \cos^2\theta\cos\phi)$$

$$= \sin^2\phi + \cos^2\phi = 1$$

$$R^{-1} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

Thus,

$$\hat{i} = \sin\theta\cos\phi\;\hat{r} + \cos\theta\cos\phi\;\hat{\theta} - \sin\phi\;\hat{\phi}$$

$$\hat{j} = \sin\theta\sin\phi\;\hat{r} + \cos\theta\sin\phi\;\hat{\theta} + \cos\phi\;\hat{\phi}$$

$$\hat{k} = \cos\theta\;\hat{r} - \sin\theta\;\hat{\theta} + 0\;\hat{\phi}$$

Prob 3 (b) : Griffiths 1.39

(a) $\int_V d^3x \; \vec{\nabla}\cdot\vec{v}_1 = \oint d\vec{a}\cdot\vec{v}_1$

$$\text{L.H.S.} = \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \left[\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \right]\cdot(r^2\hat{r})$$

$$= \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \; \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta \, r^2\right)$$

$$= \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \; 4r$$

$$= 4\pi \int_0^R dr \; 4r^3 = 4\pi R^4$$

$$\text{R.H.S.} = 4\pi R^2 \; \hat{R}\cdot(R^2\hat{R}) = 4\pi R^4 \checkmark$$

(b) $$\text{L.H.S.} = \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \; \overbrace{\frac{1}{r^2\sin\theta} \underbrace{\frac{\partial}{\partial r}\left(r^2\sin\theta \, \frac{1}{r^2}\right)}_{=0}}^{\vec{\nabla}\cdot\vec{v}_2}$$

$$= 0$$

$$\text{R.H.S.} = 4\pi R^2 \; \hat{R}\cdot\left(\frac{\hat{R}}{R^2}\right) \rightarrow \vec{v}_2 \text{ on the surface.}$$

$$= 4\pi$$

There seems to be a contradiction. This apparent breakdown is because $\vec{\nabla}\cdot\vec{v}_2$ is divergent at $r=\infty$.

The resolution is to interpret

$$\vec{\nabla}\cdot\vec{v}_2 = \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi \, \delta^{(3)}(\vec{r})$$

Problem (3) (c) : Griffiths 1.40

$$\vec{V} = r\cos\theta\,\hat{r} + r\sin\theta\,\hat{\theta} + r\sin\theta\cos\phi\,\hat{\phi}$$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\, r\cos\theta\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta\, r\sin\theta\right) + \frac{\partial}{\partial\phi}\left(r\, r\sin\theta\cos\phi\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[3r^2\sin\theta\cos\theta + r^2\, 2\sin\theta\cos\theta - r^2\sin\theta\sin\phi\right]$$

$$= 3\cos\theta + 2\cos\theta - \sin\phi$$

$$= 5\cos\theta - \sin\phi$$

$$\int_{\substack{\text{inverted}\\\text{bowl}}} d^3x\ \vec{\nabla}\cdot\vec{V} = \int_0^R r^2 dr \int_0^{\pi/2}\sin\theta\, d\theta \int_0^{2\pi} d\phi\ (5\cos\theta - \sin\phi)$$

$$= \frac{R^3}{3}\left[10\pi\int_1^0 (-dt)\, t - \frac{\pi}{2}\underbrace{\int_0^{2\pi} d\phi\,\sin\phi}_{=0}\right]$$

$$= \frac{5\pi R^3}{3}.$$

$$\oint_{\substack{\text{inverted}\\\text{bowl}}} d\vec{a}\cdot\vec{V} = \int_0^{\pi/2}\sin\theta\, d\theta\int_0^{2\pi} d\phi\ R^2\hat{R}\cdot\left[R\cos\theta\,\hat{R} + R\sin\theta\,\hat{\theta} + R\sin\theta\cos\phi\,\hat{\phi}\right]$$
$$+ \int_0^R r\,dr\int_0^{2\pi} d\phi\,(-\hat{k})\cdot\left[r\cos\theta\,\hat{r} + r\sin\theta\,\hat{\theta} + r\sin\theta\cos\phi\,\hat{\phi}\right]_{\theta=\frac{\pi}{2}}$$

$$= 2\pi R^3\int_0^{\pi/2}\sin\theta\, d\theta\,\cos\theta$$
$$- \int_0^R r\,dr\int_0^{2\pi} d\phi\left[r\cos\theta\cos\theta + r\sin\theta(-\sin\theta) + r\sin\theta\cos\phi\ 0\right]_{\theta=\frac{\pi}{2}}$$

using $\hat{r}\cdot\hat{k} = \cos\theta$, $\hat{\theta}\cdot\hat{k} = -\sin\theta$, $\hat{\phi}\cdot\hat{k} = 0$ see page (6).

$$= 2\pi R^3\,\frac{1}{2} - \int_0^R r\,dr\int_0^{2\pi} d\phi\,\left[0 - r + 0\right]$$

$$1 + \frac{2}{3} = \frac{5}{3}$$

$$= \pi R^3 + 2\pi\frac{R^3}{3}$$

$$= \frac{5\pi R^3}{3}\ \checkmark$$

Problem ③ (d) : Griffiths 1.62 (part (a))

$$\vec{a} = \int_S d\vec{a}$$

$$= R^2 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi \quad \hat{R}$$

$$= R^2 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi \left[\sin\theta \cos\phi \, \hat{i} + \sin\theta \sin\phi \, \hat{j} + \cos\theta \, \hat{k} \right]$$

$$= R^2 \hat{i} \int_0^{\pi/2} \sin^2\theta \, d\theta \underbrace{\int_0^{2\pi} d\phi \cos\phi}_{=0} + R^2 \hat{j} \int_0^{\pi/2} \sin^2\theta \, d\theta \underbrace{\int_0^{2\pi} d\phi \sin\phi}_{=0} + R^2 \hat{k} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \hat{k} \; 2\pi R^2 \int_0^{\pi/2} \sin\theta \, d\theta \cos\theta .$$

$$= \pi R^2 \hat{k}$$

Prob. (3) (e)

Using page (6) of this solution set we have.

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\,\hat{k}$$

$$\frac{\partial}{\partial\phi}\hat{\phi} = \frac{\partial}{\partial\phi}\left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right]$$

$$= -\cos\phi\,\hat{i} - \sin\phi\,\hat{j}$$

Using page (7) of this solution set

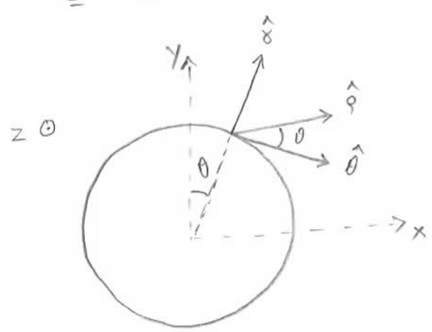
$$\hat{i} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi},$$

$$\hat{j} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi},$$

which when used in $\frac{\partial}{\partial\phi}\hat{\phi}$ leads to

$$\frac{\partial}{\partial\phi}\hat{\phi} = -\cos\phi\left[\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right]$$
$$-\sin\phi\left[\sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}\right]$$

$$= -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta} + 0\,\hat{\phi}.$$



$$\hat{\rho} = \cos\theta\,\hat{\theta} + \sin\theta\,\hat{r}$$
$$= -\frac{\partial}{\partial\phi}\hat{\phi}$$

Prob (4) (a) : Griffiths 1.44

(a) $\int_2^6 dx\ \delta(x-3)\ [3x^2 - 2x - 1]$

$= 3(3)^2 - 2(3) - 1 = 20$

(b) $\int_0^5 dx\ \delta(x-\pi)\ \cos x$ $\qquad 0 < (\pi \sim 3.14) < 5$

$= \cos\pi = -1$

(c) $\int_0^3 dx\ \delta(x+1)\ x^3$ $\qquad -1 < 0$

$= 0$ $\qquad$ δ-function does not contribute.

(d) $\int_{-\infty}^{+\infty} dx\ \delta(x+2)\ \ln(x+3)$

$= \ln(-2+3) = 0.$

Prob 4 (b) : Griffiths 1.45

(a) $\int_{-2}^{2} dx\ \delta(3x)\ [2x+3]$

$$= \int_{-2}^{+2} dx\ \frac{1}{3}\ \delta(x)\ [2x+3]$$

$$= \frac{1}{3}\ [2 \cdot 0 + 3] = 1.$$

(b) $\int_{0}^{2} dx\ \delta(1-x)\ [x^3 + 3x + 2]$

$$= \int_{0}^{2} dx\ \delta(-(x-1))\ [x^3 + 3x + 2]$$

$$= \int_{0}^{2} dx\ \frac{1}{|-1|}\ \delta(x-1)\ [x^3 + 3x + 2]$$

$$= 1^3 + 3 \cdot 1 + 2 = 6.$$

(c) $\int_{-1}^{+1} dx\ \delta(3x+1)\ 9x^2$

$$= \int_{-1}^{+1} dx\ \frac{1}{3}\ \delta(x + \tfrac{1}{3})\ 9x^2$$

$$= \frac{1}{3}\ 9 \left(-\frac{1}{3}\right)^2 = \frac{1}{3}.$$

(d) $$\int_{-\infty}^{a} dx\ \delta(x-b) = \begin{cases} 1 & \text{if } b < a. \\ 0 & \text{if } b > a \end{cases}$$

Prob 4 (c) : Griffiths 1.47

(a) $$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r}-\vec{r}')$$

$$\int d^3\vec{r}\;\rho(\vec{r}) = q\int d^3r\;\delta^{(3)}(\vec{r}-\vec{r}') = q.$$

(b) $$\rho(\vec{r}) = -q\,\delta^{(3)}(\vec{r}) + q\,\delta^{(3)}(\vec{r}-\vec{a})$$

$$\int d^3r\;\rho(\vec{r}) = 0$$

(c) $$\rho(\vec{r}) = \frac{Q}{4\pi R^2}\,\delta(r-R)$$

$$\int d^3r\;\rho(\vec{r}) = \int_0^\infty r^2\,dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\phi\;\frac{Q}{4\pi R^2}\,\delta(r-R)$$

$$= R^2\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\phi\;\frac{Q}{4\pi R^2}$$

$$= Q.$$

Prob (4) (d) : Griffiths 1.48

(a) $\int d^3r \; \delta^{(3)}(\vec{r}-\vec{a}) \; [r^2 + \vec{r}\cdot\vec{a} + a^2]$

$= a^2 + \vec{a}\cdot\vec{a} + a^2 = 3a^2$

(b) $\int_V d^3r \; \delta^{(3)}(5\vec{r}) \; |\vec{r}-\vec{b}|^2$

$\vec{r} = \vec{0}$ is inside the cube.

$= \int_V d^3r \left(\frac{1}{5}\right)^3 \delta^{(3)}(\vec{r}) \; |\vec{r}-\vec{b}|^2$

$\vec{b} = 4\hat{j} + 3\hat{k}$

$= \left(\frac{1}{5}\right)^3 |\vec{b}|^2$

$|\vec{b}| = 5$

$= \frac{1}{5}$.

V: sphere of radius 6.

$\vec{c} = 5\hat{i} + 3\hat{j} + 2\hat{k}$

$|\vec{c}| = \sqrt{25+9+4} = \sqrt{38} > 6$

(c) $\int_V d^3r \; \delta^{(3)}(\vec{r}-\vec{c}) \; [r^4 + r^2(\vec{r}\cdot\vec{c}) + c^4]$

$= 0$,

because $\vec{c}$ is outside the region of integration.

V: sphere of radius 1.5 centered at (2,2,2).

(d) $\int_V d^3r \; \delta^{(3)}(\vec{e}-\vec{r}) \; \vec{r}\cdot(\vec{d}-\vec{r})$

$\vec{e} = (3,2,1)$

$\vec{e}$ is inside V.

$= \vec{e}\cdot(\vec{d}-\vec{e})$

$\vec{d} = (1,2,3)$

$= (3\hat{i} + 2\hat{j} + \hat{k}) \cdot [-2\hat{i} + 0\hat{j} + 2\hat{k}]$

$\vec{d}-\vec{e} = -2\hat{i} + 0\hat{j} + 2\hat{k}$

$= -6 + 0 + 2$

$= -4$.