Homework No. 03 (Fall 2013)

PHYS 320: Electricity and Magnetism I

Due date: Friday, 2013 Sep 27, 4.30pm

1. For a wire of negligible cross section, any volume integral involving the current density $\mathbf{j}$ becomes a line integral
$$\int d^3r\,\mathbf{j} = \int d\mathbf{l}\,I, \tag{1}$$
after one identifies the current density as the charge flux vector for the current
$$I = \int d\mathbf{a}\cdot\mathbf{j}. \tag{2}$$
Deduce the relation
$$\mathbf{j} = \rho\mathbf{v}, \tag{3}$$
where ρ is the charge density and $\mathbf{v} = d\mathbf{l}/dt$ is the velocity of the charge flowing in the wire.

2. The relation between charge density and current density,
$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) = 0, \tag{4}$$
is the general statement of the conservation of charge. For an arbitrarily moving point particle with charge, the charge and current densities are
$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{r}_a(t)) \tag{5}$$
and
$$\mathbf{j}(\mathbf{r},t) = q\mathbf{v}_a(t)\,\delta(\mathbf{r}-\mathbf{r}_a(t)), \tag{6}$$
where $\mathbf{r}_a(t)$ is the position vector and
$$\mathbf{v}_a(t) = \frac{d\mathbf{r}_a}{dt} \tag{7}$$
is the velocity of the charged particle. Verify the statement of the conservation of charge in Eq. (4) for a point particle.

3. The electric and magnetic fields are defined in terms of the scalar and vector potentials by the relations
$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial t}\mathbf{A}. \tag{8}$$

Show that the potentials are not uniquely defined in that if we let

$$\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\lambda, \qquad \phi \to \phi - \frac{\partial}{\partial t}\lambda, \tag{9}$$

the electric and magnetic fields in Eq. (8) remain unaltered, for an arbitrary function $\lambda = \lambda(\mathbf{r}, t)$. This is called gauge invariance and the corresponding substitution in Eq. (9) is a gauge transformation.

4. Problem 2.46 in Griffiths 4th edition.

5. Consider a line segment of length $2L$ with uniform line charge density λ.

   (a) When the rod is placed on the z-axis centered on the origin, show that the charge density can be expressed as

   $$\rho(\mathbf{r}) = \lambda\delta(x)\delta(y)f(z), \tag{10}$$

   where $f(z) = 0$, if $z > L$ and $z < -L$, and $f(z) = 1$, otherwise.

   (b) Evaluate the electric potential of the rod as

   $$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[\sinh^{-1}\left(\frac{L-z}{\sqrt{x^2+y^2}}\right) + \sinh^{-1}\left(\frac{L+z}{\sqrt{x^2+y^2}}\right)\right]. \tag{11}$$

   (c) Show that

   $$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}). \tag{12}$$

   (d) Thus, express the electric potential of Eq. (11) in the form

   $$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[-2\ln\frac{R}{L} + F\left(\frac{z}{L}, \frac{R}{L}\right)\right], \tag{13}$$

   where $R^2 = x^2 + y^2$ and

   $$F(a,b) = \ln[1 - a + \sqrt{(1-a)^2 + b^2}] + \ln[1 + a + \sqrt{(1+a)^2 + b^2}]. \tag{14}$$

   (e) Show that

   $$\phi(\mathbf{r}) \xrightarrow{R \ll L, z \ll L} -\frac{2\lambda}{4\pi\varepsilon_0}\ln\frac{R}{2L}. \tag{15}$$

   (f) Using $\mathbf{E} = -\boldsymbol{\nabla}\phi$ determine the electric field for an infinite rod (placed on the z-axis).

   (g) What can you conclude about the electric field due to a point charge if our space was two dimensional?