Homework No. 06 (Fall 2013)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2013 Nov 13, 4.30pm

1. Verify the identity

$$\phi \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} \psi) - \psi \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} \phi) = \boldsymbol{\nabla} \cdot [\lambda (\phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi)], \tag{1}$$

which is a slight generalization of what is known as Green's second identity.

2. Show that

$$\bar{\delta}(x) = -x \frac{d}{dx} \delta(x) \tag{2}$$

is also a model of the δ-function by showing that

$$\int_{-\infty}^{\infty} dx \, \bar{\delta}(x) f(x) = f(0). \tag{3}$$

3. A forced harmonic oscillator is described by the differential equation

$$-\left(\frac{d^2}{dt^2} + \omega^2 \right) x(t) = F(t), \tag{4}$$

where ω is the angular frequency of the oscillator and $F(t)$ is the forcing function. The corresponding Green's function satisfies

$$-\left(\frac{d^2}{dt^2} + \omega^2 \right) G(t, t') = \delta(t - t'). \tag{5}$$

The continuity conditions satisfied by the Greens function are

$$\frac{d}{dt} G(t, t') \bigg|_{t=t'-\delta}^{t=t'+\delta} = -1 \tag{6}$$

and

$$G(t, t') \bigg|_{t=t'-\delta}^{t=t'+\delta} = 0. \tag{7}$$

(a) Verify that a particular solution,

$$G_R(t - t') = -\frac{1}{\omega} \theta(t - t') \sin \omega (t - t'), \tag{8}$$

which is referred to as the retarded Green's function, satisfies the Greens function differential equation and the continuity conditions.

(b) Verify that another particular solution,

$$G_A(t - t') = \frac{1}{\omega}\theta(t' - t)\sin\omega(t - t'), \tag{9}$$

which is referred to as the advanced Green's function, satisfies the Greens function differential equation and the continuity conditions.

(c) Show that the difference of the two particular solutions above,

$$G_R(t - t') - G_A(t - t'), \tag{10}$$

satisfies the homogeneous differential equations

$$-\left(\frac{d^2}{dt^2} + \omega^2\right) G_0(t, t') = 0. \tag{11}$$

4. The electric potential is given in terms of the Greens function by the expression

$$\phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int d^3r' G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}'). \tag{12}$$

A representation of the Greens function that is suitable for the case when the charge density is a function of z alone is

$$G(\mathbf{r}, \mathbf{r}') = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} \frac{1}{2k_\perp} e^{-k_\perp|z-z'|}. \tag{13}$$

(a) Express the electric potential due to an infinitely thin plate described by the charge density $\rho(\mathbf{r}) = \sigma\delta(z - a)$ in the form

$$\phi(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0}\lim_{k_\perp\to 0}\left[\frac{1}{k_\perp} - |z - a|\right]. \tag{14}$$

Hint: Start by evaluating the z' integral, that involves a δ-function integral, after substituting the expressions for $G(\mathbf{r}, \mathbf{r}')$ and $\rho(\mathbf{r}')$ into Eq. (12). Use the δ-function representation,

$$\int_{-\infty}^{\infty} dx\, e^{ikx} = 2\pi\delta(k), \tag{15}$$

to complete the integrations on x' and y'. Then complete the k_x and k_y integral in the form of limits after expanding the exponential using Taylor series.

(b) Show that the electric field then is

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = \begin{cases} \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}, & z > a, \\ -\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}, & z < a, \end{cases} \tag{16}$$