Homework No. 07 (Fall 2013)

PHYS 320: Electricity and Magnetism I

Due date: Monday, 2013 Dec 2, 4.30pm

1. Find the solution to the differential equation

$$\left[-\frac{\partial}{\partial z}\varepsilon(z)\frac{\partial}{\partial z}+\varepsilon(z)k_\perp^2\right]g(z,z';k_\perp)=\delta(z-z') \tag{1}$$

   when

$$\varepsilon(z)=\begin{cases}\varepsilon_2 & z<a,\\ \varepsilon_1 & a<z.\end{cases} \tag{2}$$

   for the case $a<z'$. Look for solution that is zero at $z=\pm\infty$.

2. Consider a semi-infinite dielectric slab described by

$$\varepsilon(z)=\begin{cases}\varepsilon_2 & z<a,\\ \varepsilon_1>\varepsilon_2 & a<z.\end{cases} \tag{3}$$

   A point charge q described by

$$\rho(\mathbf{r})=q\delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{4}$$

   is embedded at position $\mathbf{r}'$ (with $a<z'$) on one side of the interface.

   (a) Show that the electric potential is given in terms of the Green's function by

$$\phi(\mathbf{r})=qG(\mathbf{r},\mathbf{r}'), \tag{5}$$

   where the Green's function satisfies

$$\boldsymbol{\nabla}\cdot\varepsilon(z)\boldsymbol{\nabla}G(\mathbf{r},\mathbf{r}')=\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{6}$$

   Using the solution for the reduced Green's function $g(z,z';k_\perp)$ find the expression for the electric potential to be given by

$$\phi(\mathbf{r})=\begin{cases}\dfrac{q}{4\pi\varepsilon_1}\dfrac{1}{|\mathbf{r}-\mathbf{r}'|}+\dfrac{q}{4\pi\varepsilon_1}\dfrac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\dfrac{1}{|\mathbf{r}-\mathbf{r}'_{\text{im}}|}, & a<z,\\[2ex] \dfrac{2q}{4\pi(\varepsilon_1+\varepsilon_2)}\dfrac{1}{|\mathbf{r}-\mathbf{r}'|}, & z<a,\end{cases} \tag{7}$$

   where $\mathbf{r}'_{\text{im}}=\mathbf{r}'-2(z'-a)\hat{\mathbf{z}}$.

(b) Using $\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r})$ find the expression for the electric field as

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\varepsilon_1}\dfrac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} + \dfrac{q}{4\pi\varepsilon_1}\dfrac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\dfrac{\mathbf{r}-\mathbf{r}'_{\text{im}}}{|\mathbf{r}-\mathbf{r}'_{\text{im}}|^3}, & a < z, \\ \dfrac{2q}{4\pi(\varepsilon_1+\varepsilon_2)}\dfrac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, & z < a. \end{cases} \tag{8}$$

(c) Draw the electric field lines for this configuration ($\varepsilon_2 < \varepsilon_1$).

(d) Investigate the continuity in the components of electric field at the interface by evaluating the following:

$$E_x(x, y, a+\delta) - E_x(x, y, a-\delta) = ?, \tag{9}$$

$$E_y(x, y, a+\delta) - E_y(x, y, a-\delta) = ?, \tag{10}$$

$$\varepsilon_1 E_z(x, y, a+\delta) - \varepsilon_2 E_z(x, y, a-\delta) = ?. \tag{11}$$