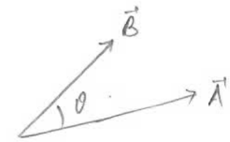
① Vectors (in 3-D)

$$\vec{A} = (A_1, A_2, A_3) = A_1 \hat{x} + A_2 \hat{y} + A_3 \hat{z}.$$

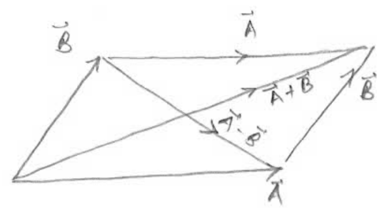
$$\vec{C} = \vec{A} \pm \vec{B}$$ addition & subtraction



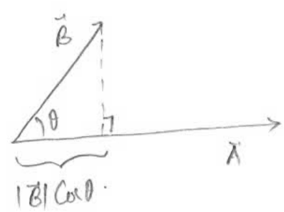
$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$
$$= |\vec{A}|\,|\vec{B}| \cos\theta.$$

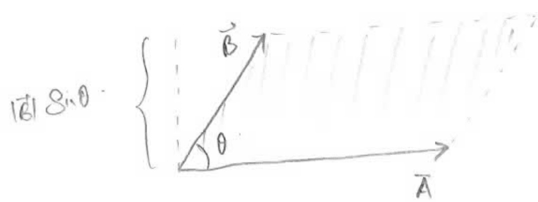
$$\vec{A} \times \vec{B} = \hat{n}\, |\vec{A}|\,|\vec{B}| \sin\theta.$$
$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

② Illustrations and interpretations



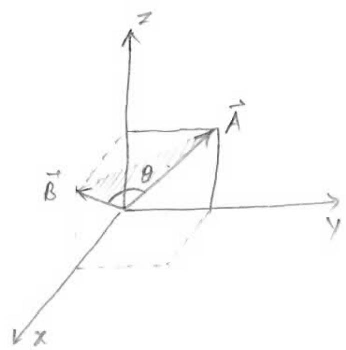
addition and subtraction





Area $= |\vec{A} \times \vec{B}|$

3 Example: Find the angle between two face diagonals of a cube (originating from origin).



$\vec{A} = (0, 1, 1)$ $\quad |\vec{A}| = \sqrt{2}$

$\vec{B} = (1, 0, 1)$ $\quad |\vec{B}| = \sqrt{2}$

$\vec{A} \cdot \vec{B} = 1$

$$\vec{A} \cdot \vec{B} = |\vec{A}|\, |\vec{B}| \cos\theta$$

$$1 = \sqrt{2}\, \sqrt{2} \cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 60^\circ$$

(4) Linear transformation of a vector

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\downarrow$ transformed vector; $\downarrow$ Transformation matrix; $\downarrow$ original vector

$$x_1' = R_{11} x_1 + R_{12} x_2 + R_{13} x_3$$

$$= \sum_{j=1}^{3} R_{1j} x_j$$

$$= R_{1j} x_j$$

(Summation convention, repeated index implies sum)

$$x_i' = R_{ij} x_j \qquad i = 1, 2, 3$$

$i \to$ free index

$j \to$ dummy index

(5) A rotation transformation keeps the following physical quantities invariant:

(i) Length of a vector

(ii) Angle between two vectors.

(6) Let us determine the transformation that keeps the distance invariant.

$$\vec{x} \cdot \vec{x} = x_i x_i = x'_i x'_i$$
$$= R_{ij} x_j \; R_{ik} x_k$$
$$= R_{ij} R_{ik} \; x_j x_k.$$

(7) $$\delta_{ij} \rightarrow \overleftrightarrow{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i = j \end{cases}$$

(8) $$x_i x_i = x_i \delta_{ij} x_j$$
$$= \delta_{ij} \; x_i x_j$$
$$= \delta_{jk} \; x_j x_k$$ (trick involving dummy indices)

(9) Using (6) & (8) → Introduce an example to illustrate multiplication of matrices using index notation.

$$R_{ij} R_{ik} = \delta_{jk}$$

(10) Comparing with

$$\overleftrightarrow{R}^{-1} \cdot \overleftrightarrow{R} = \overleftrightarrow{1}$$

or $$(R^{-1})_{ji} \; R_{ik} = \delta_{jk}.$$

$$\overleftrightarrow{R} \rightarrow R_{ij}$$

$$\overleftrightarrow{R}^{T} \rightarrow R_{ji}$$

$\downarrow$

transpose of $\overleftrightarrow{R}$.

$$\Rightarrow \quad (R^{-1})_{ji} = R_{ij}$$

$$\Rightarrow \quad \overleftrightarrow{R}^{-1} = \overleftrightarrow{R}^{T}$$

11 Thus, the transformation that keeps the length of a vector invariant satisfies

$$\overleftrightarrow{R}^{-1} = \overleftrightarrow{R}^{T},$$

and are called orthogonal transformations.

12 Since

$$\overleftrightarrow{R}^{T} \cdot \overleftrightarrow{R} = \overleftrightarrow{I}$$

$$\det(\overleftrightarrow{R}^{T} \cdot \overleftrightarrow{R}) = \det(\overleftrightarrow{I})$$

$$\det(\overleftrightarrow{R}^{T}) \det(\overleftrightarrow{R}) = 1 \qquad (\det(\overleftrightarrow{A}\cdot\overleftrightarrow{B}) = (\det \overleftrightarrow{A})(\det \overleftrightarrow{B}))$$

$$(\det \overleftrightarrow{R})^2 = 1 \qquad (\det \overleftrightarrow{R}^{T} = \det \overleftrightarrow{R})$$

$$\det \overleftrightarrow{R} = \pm 1$$

13 2-dimensions — an example.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

all real number

$$R_{ij} R_{ik} = \delta_{jk}$$

$j=1 \quad k=1 \qquad R_{i1} R_{i1} = \delta_{11} \Rightarrow R_{11} R_{11} + R_{21} R_{21} = 1 \Rightarrow \alpha^2 + \gamma^2 = 1$

$j=1 \quad k=2 \qquad R_{i1} R_{i2} = \delta_{12} \Rightarrow R_{11} R_{12} + R_{21} R_{22} = 0 \Rightarrow \alpha\beta + \gamma\delta = 0$

$j=2 \quad k=1 \qquad R_{i2} R_{i1} = \delta_{21} \Rightarrow R_{12} R_{11} + R_{22} R_{21} = 0$

$j=2 \quad k=2 \qquad R_{i2} R_{i2} = \delta_{22} \Rightarrow R_{12} R_{12} + R_{22} R_{22} \Rightarrow \beta^2 + \delta^2 = 1$

Further, $\det \overleftrightarrow{R} = \pm 1 \Rightarrow R_{11} R_{22} - R_{21} R_{12} = \pm 1 \Rightarrow \alpha\delta - \beta\gamma = \pm 1$

(14) Let $\alpha = \cos\theta_1 \Rightarrow \gamma = \pm \sin\theta_1$

$\delta = \cos\theta_2 \Rightarrow \beta = \pm \sin\theta_2$.

$\alpha\beta + \gamma\delta = 0 \Rightarrow \cos\theta_1 \sin\theta_2 \pm \cos\theta_2 \sin\theta_1 = 0$

$\tan\theta_1 = \pm \tan\theta_2$.

$\Rightarrow \theta_1 = \pm \theta_2$.

(15) Proper rotation $\longrightarrow$ $\det \overleftrightarrow{R} = +1$ (rotation, inversion in 2-D)

Improper rotation $\longrightarrow$ $\det \overleftrightarrow{R} = -1$ (reflection, inversion in 3-D)

(16) Quantities that transform like a position vector is called a vector

scalar $\rightarrow$ $f(x_i') = f(x_i)$

vector $\rightarrow$ $A_i' = R_{ij} A_j$

tensor $\rightarrow$ $A_{ij} = R_{im} R_{jn} A_m A_n$

n-th rank tensor $\rightarrow$ $A'_{i_1 i_2 \dots i_n} = R_{i_1 i_1'} R_{i_2 i_2'} \dots R_{i_n i_n'} A_{i_1'} A_{i_2'} \dots A_{i_n'}$

17 Kronecker delta $\rightarrow$ $$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

Levi-Civita symbol $\rightarrow$ $$\epsilon_{ijk} = \begin{cases} +1 & \text{if} \quad 123 \quad \text{and even permutations} \\ -1 & \text{if} \quad 132 \quad \text{and even permutations} \\ 0 & \text{otherwise} \end{cases}$$

18 An identity:

$$\epsilon_{ijk}\,\epsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$$

19 Also note

$$\delta_{ii} = 3$$

21 Setting $k = r$ in 18

$$\begin{aligned} \epsilon_{ijk}\,\epsilon_{pqk} &= \delta_{ip}(\delta_{jq}\delta_{kk} - \delta_{jk}\delta_{kq}) - \delta_{iq}(\delta_{jp}\delta_{kk} - \delta_{jk}\delta_{kp}) \\ &\quad + \delta_{ik}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) \\ &= 3\,\delta_{ip}\delta_{jq} - \delta_{ip}\delta_{jq} - 3\,\delta_{iq}\delta_{jp} + \delta_{iq}\delta_{jp} \\ &\quad + \delta_{iq}\delta_{jp} - \delta_{jq}\delta_{ip} \\ &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \end{aligned}$$

22 Setting $j = q$ in 21

$$\epsilon_{ijk}\,\epsilon_{pjk} = \delta_{ip}\,\delta_{jj} - \delta_{ij}\,\delta_{jp}$$
$$= 3\,\delta_{ip} - \delta_{ip}$$
$$= 2\,\delta_{ip}$$

23 Setting $i = p$ in 22

$$\epsilon_{ijk}\,\epsilon_{ijk} = 2\,\delta_{ii}$$
$$= 6$$

24
$$\vec{A}\cdot\vec{B} = A_i\,B_i$$
$$\vec{A}\times\vec{B} = \epsilon_{ijk}\,A_j\,B_k$$
$$Tr\,(\overleftrightarrow{M}) = M_{ii}$$
$$\det\,(\overleftrightarrow{M}) = \epsilon_{ijk}\,M_{i1}\,M_{j2}\,M_{k3}$$
$$= \frac{1}{3!}\,\epsilon_{ijk}\,\epsilon_{i'j'k'}\,M_{ii'}\,M_{jj'}\,M_{kk'}$$

25 $$\vec{A} \times (\vec{B} \times \vec{C}) = \epsilon^{ijk} A^j (\vec{B} \times \vec{C})^k$$
$$= \epsilon^{ijk} A^j \epsilon^{kmn} B^m C^n$$
$$= \epsilon^{ijk} \epsilon^{kmn} A^j B^m C^n$$
$$= (\delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}) A^j B^m C^n$$
$$= A^j B^i C^j - A^j B^j C^i$$
$$= \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$$

26 $$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = \epsilon^{ijk} A^j B^k \epsilon^{imn} C^m D^n$$
$$= \epsilon^{ijk} \epsilon^{imn} A^j B^k C^m D^n$$
$$= (\delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}) A^j B^k C^m D^n$$
$$= A^j B^k C^j D^k - A^j B^k C^k D^j$$
$$= (\vec{A} \cdot \vec{C}) (\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D}) (\vec{B} \cdot \vec{C})$$