SET-03

Vector calculus (Differentiation)

(1) Variation in 1-D

$f = f(x)$ $\qquad df = dx \frac{d}{dx} f$

(2) Variation in 3-D

$f = f(x, y, z)$ $\qquad df = dx \frac{\partial}{\partial x} f + dy \frac{\partial}{\partial y} f + dz \frac{\partial}{\partial z} f$

(3) Partial derivative – illustrative example

$$f(x, y, z) = x^3 y^2 + x \sin y + z$$

$$\frac{\partial}{\partial x} f = 3x^2 y^2 + \sin y$$

$$\frac{\partial}{\partial y} f = 2x^3 y + x \cos y$$

$$\frac{\partial}{\partial z} f = 1$$

(4) We can write

$$df = dx \frac{\partial}{\partial x} f + dy \frac{\partial}{\partial y} f + dz \frac{\partial}{\partial z} f$$

$$= d\vec{l} \cdot \vec{\nabla} f$$

where

$$d\vec{l} = dx\, \hat{x} + dy\, \hat{y} + dz\, \hat{z}$$

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

(5) Example :

$\gamma = \sqrt{x^2 + y^2 + z^2}$

(i) $\vec{\nabla} \gamma = \vec{\nabla} \sqrt{x^2 + y^2 + z^2}$

$= \hat{x} \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + x^2} + \hat{y} \frac{\partial}{\partial y} \sqrt{x^2 + y^2 + z^2} + \hat{z} \frac{\partial}{\partial z} \sqrt{x^2 + y^2 + z^2}$

$= \frac{x \hat{x} + y \hat{y} + z \hat{z}}{\gamma}$ $\qquad \vec{\gamma} = x \hat{x} + y \hat{y} + z \hat{z}$

$= \frac{\vec{\gamma}}{\gamma} = \hat{\gamma}$

(ii) $\vec{\nabla} \gamma^n = n \gamma^{n-1} \vec{\nabla} \gamma$

$= n \gamma^{n-1} \hat{\gamma}$

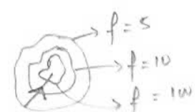
Griffiths defines.

$\vec{\mathscr{r}} = \vec{\gamma} - \vec{\gamma}'$

(iii) $\vec{\nabla} |\vec{\gamma} - \vec{\gamma}'|^n = n |\vec{\gamma} - \vec{\gamma}'|^{n-1} \frac{(\vec{\gamma} - \vec{\gamma}')}{|\vec{\gamma} - \vec{\gamma}'|}$

(6) Interpretation :

(i) $\vec{\nabla} f$ is normal to the curve/surface, f = constant, in 2-D/3-D.



(ii) $\vec{\nabla} f$ points in the direction of maximum change in f, and its magnitude is the slope at the point.

→ $x^2 + y^2 + z^2 = \gamma^2$ describes surface of a sphere, and $\vec{\nabla}(x^2 + y^2 + z^2) = 2\vec{\gamma}$ is normal to this surface.

→ $x^2 + y^2 - z^2 \tan^2\theta = 0$ describes a surface of cone, constant θ,
$\vec{\nabla}(x^2 + y^2 - z^2 \tan^2\theta) = 2x\hat{x} + 2y\hat{y} - 2z\tan^2\theta\, \hat{z}$ normal to the cone.
points in the direction of $\hat{\theta}$,

7. <u>Example</u> (Griffiths 4e, problem 1.12)

Height of a hill is given by

$$h(x, y) = 10\left[2xy - 3x^2 - 4y^2 - 18x + 28y + 12\right]$$

(i) where is top of hill located?

$$\vec{\nabla} h = 10(2y - 6x - 18)\,\hat{i} + 10(2x - 8y + 28)\,\hat{j}$$

$$\vec{\nabla} h = 0 \Rightarrow x = -2,\ y = 3 \leftarrow \text{location of top}$$

(ii) how high is the hill?

$$h(-2, 3) = 720 \text{ miles.}$$

8. <u>Divergence</u> of a vector field.

$$\vec{\nabla} \cdot \vec{A} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left[\hat{i} A_x + \hat{j} A_y + \hat{k} A_z\right]$$

$$= \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

interpretation — sum of magnitudes of three derivatives of the field $\vec{A}$,
— a measure of divergence of a field.
source and sink of a field.

9. <u>Curl</u> of a vector field.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y\right) - \hat{j}\left(\frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x\right) + \hat{k}\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right)$$

interpretation — a measure of vorticity of the field,
example how much the water is spinning inside a sink.

(10) Example

$$\vec{\nabla}\,\vec{r} = \nabla^i x^j = \frac{\partial}{\partial x^i} x^j = \delta_{ij} = \overleftrightarrow{1}$$

$\Rightarrow$

$$\vec{\nabla}\cdot\vec{r} = \delta_{ii} = 3$$

$$\vec{\nabla}\times\vec{r} = \epsilon^{ijk}\delta_{jk} = \epsilon^{ijj} = 0$$

(11) Example

(i)
$$\vec{\nabla}\,\frac{\vec{r}}{r} = (\vec{\nabla}\,\vec{r})\frac{1}{r} + \vec{r}\,\vec{\nabla}\frac{1}{r} = \overleftrightarrow{1}\,\frac{1}{r} - \vec{r}\,\frac{1}{r^2}\,\vec{\nabla} r = \overleftrightarrow{1}\,\frac{1}{r} - \frac{\vec{r}\,\vec{r}}{r^3}$$

$\Rightarrow$

$$\vec{\nabla}\cdot\frac{\vec{r}}{r} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\vec{\nabla}\times\frac{\vec{r}}{r} = 0$$

(ii)
$$\vec{\nabla}\,\frac{\vec{r}}{r^2} = (\vec{\nabla}\,\vec{r})\frac{1}{r^2} + \vec{r}\,\vec{\nabla}\frac{1}{r^2} = \overleftrightarrow{1}\,\frac{1}{r^2} - 2\,\frac{\vec{r}}{r^3}\,\vec{\nabla} r = \overleftrightarrow{1}\,\frac{1}{r^2} - 2\,\frac{\vec{r}\,\vec{r}}{r^4}$$

$\Rightarrow$

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^2} = \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

$$\vec{\nabla}\times\frac{\vec{r}}{r^2} = 0$$

(iii)
$$\vec{\nabla}\,\frac{\vec{r}}{r^3} = \overleftrightarrow{1}\,\frac{1}{r^3} - 3\,\frac{\vec{r}\,\vec{r}}{r^5}$$

$\Rightarrow$

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} \stackrel{?}{=}$$

$$\vec{\nabla}\times\frac{\vec{r}}{r^3} = 0$$

(12) Laplacian

$$\vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

(13) $\vec{\nabla} \times \vec{\nabla} f = 0$

$$\vec{\nabla} \times \vec{\nabla} = \epsilon^{ijk} \nabla^j \nabla^k$$

$\begin{cases} \nabla^j \nabla^k - \text{symmetric in } jk \\ \epsilon^{ijk} - \text{antisymmetric} \end{cases}$

$$= \tfrac{1}{2}\left[\epsilon^{ijk} \nabla^j \nabla^k + \epsilon^{ijk} \nabla^j \nabla^k \right]$$

(trick on dummy indices)

$$= \tfrac{1}{2}\left[\epsilon^{ijk} \nabla^j \nabla^k + \epsilon^{ikj} \nabla^k \nabla^j \right]$$

$(\because \nabla^k \nabla^j = \nabla^j \nabla^k)$

$$= \tfrac{1}{2}\left[\epsilon^{ijk} \nabla^j \nabla^k + \epsilon^{ikj} \nabla^j \nabla^k \right]$$
$$= \tfrac{1}{2} \left(\epsilon^{ijk} + \epsilon^{ikj} \right) \nabla^j \nabla^k$$
$$= \tfrac{1}{2} \left(\epsilon^{ijk} - \epsilon^{ijk} \right) \nabla^j \nabla^k$$
$$= 0$$

(14) $(\vec{\nabla} f) \times (\vec{\nabla} g) = \epsilon^{ijk} (\nabla^j f)(\nabla^k g)$

note that $(\nabla^j f)(\nabla^k g) \neq (\nabla^k f)(\nabla^j g)$

and thus is not symmetric in $j \leftrightarrow k$.

(15) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \epsilon^{ijk} \nabla^i \nabla^j A^k$

$= 0$ (due to conflict between symmetry and antisymmetry.)

(16) Remember:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})$$

(17)

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \epsilon^{ijk}\, \nabla^j (\epsilon^{kmn}\, \nabla^m A^n) \\
&= \epsilon^{kij}\, \epsilon^{kmn}\, \nabla^j \nabla^m A^n \\
&= (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})\, \nabla^j \nabla^m A^n \\
&= \nabla^j \nabla^i A^j - \nabla^j \nabla^j A^i \\
&= \vec{\nabla}\,(\vec{\nabla}\cdot\vec{A}) - \nabla^2 \vec{A}
\end{aligned}$$

(18)

$$\begin{aligned}
\vec{A} \times (\vec{\nabla} \times \vec{B}) &= \epsilon^{ijk}\, A^j\, \epsilon^{kmn}\, \nabla^m B^n \\
&= (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})\, A^j \nabla^m B^n \\
&= A^j \nabla^i B^j - A^j \nabla^j B^i \\
&= (\vec{\nabla}\ \vec{B})\cdot\vec{A} - \vec{A}\cdot\vec{\nabla}\ \vec{B}
\end{aligned}$$

(19)

$$\begin{aligned}
\vec{\nabla} \times (\vec{A} \times \vec{B}) &= \epsilon^{ijk}\, \nabla^j\, \epsilon^{kmn}\, A^m B^n \\
&= (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})\, \nabla^j (A^m B^n) \\
&= (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}) \left[(\nabla^j A^m)\, B^n + A^m (\nabla^j B^n) \right] \\
&= (\nabla^j A^i)\, B^j + A^i \nabla^j B^j - (\nabla^j A^j)\, B^i - A^j (\nabla^j B^i) \\
&= (\vec{B}\cdot\vec{\nabla})\,\vec{A} + \vec{A}\,(\vec{\nabla}\cdot\vec{B}) - (\vec{\nabla}\cdot\vec{A})\,\vec{B} - (\vec{A}\cdot\vec{\nabla})\,\vec{B}
\end{aligned}$$