SET – 04

Integral vector calculus

* Line integral $\longrightarrow$ $\displaystyle\int_{\substack{\vec{a}\\ \text{path}}}^{\vec{b}} d\vec{l} \cdot \vec{A}(\vec{x})$

  Surface integral $\longrightarrow$ $\displaystyle\int_S d\vec{a} \cdot \vec{A}(\vec{x})$

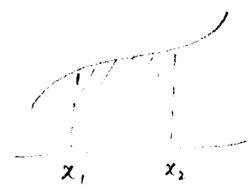
  Volume integral $\longrightarrow$ $\displaystyle\int d^3x \; f(\vec{x})$

* Theorem of gradients : $\displaystyle\int_a^{\vec{b}} d\vec{l} \cdot \vec{\nabla} f = f(\vec{b}) - f(\vec{a})$

  Theorem of curl : $\displaystyle\int_S d\vec{a} \cdot \vec{\nabla} \times \vec{A} = \oint d\vec{l} \cdot \vec{A}$

  Theorem of divergence : $\displaystyle\int_V d^3x \; \vec{\nabla} \cdot \vec{A} = \oint d\vec{a} \cdot \vec{A}$

I Review 1-D calculus



1. $$\int_{x_1}^{x_2} dx\, f(x) = \text{Area}$$

2. $$\int_{x_1}^{x_2} dx\, \frac{\partial}{\partial x} f = f(x_2) - f(x_1)$$

This can be verified in the following way

$$\int_{x_1}^{x_2} dx\, \frac{\partial}{\partial x} f = \sum_{i=1}^{N} dx \, \frac{f(x_1 + i dx) - f(x + (i-1)dx)}{dx}$$

$x_2 - x_1 = N dx$.

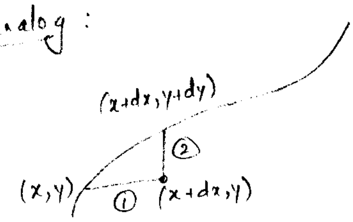
$$\begin{aligned}
&= f(x_1 + dx) - f(x_1) \\
&\quad + f(x_1 + 2dx) - f(x_1 + dx) \\
&\quad \vdots \\
&\quad + f(x_1 + N dx) - f(x_1 + (N-1)dx) \\
&= f(x_1 + N dx) - f(x_1) \\
&= f(x_2) - f(x_1)
\end{aligned}$$

II Theorem of gradients

1. $$\int_{\vec a}^{\vec b} d\vec l \cdot \vec\nabla f = f(\vec b) - f(\vec a)$$

2. $$d\vec l = \hat i\, dx + \hat j\, dy + \hat k\, dz$$
$$\vec\nabla = \hat i \frac{\partial}{\partial x} + \hat j \frac{\partial}{\partial y} + \hat k \frac{\partial}{\partial z}$$
$$d\vec l \cdot \vec\nabla = dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z}$$

3. 2-D analog:



— choose a path
— break the path into steps

$$dx \frac{\partial}{\partial x} f + dy \frac{\partial}{\partial y} f = f(x+dx, y) - f(x,y) \quad \leftarrow ①$$
$$+ f(x+dx, y+dy) - f(x+dx, y) \quad \leftarrow ②$$
$$= f(x+dx, y+dy) - f(x,y)$$

4. — N infinitesimal elements
— adjacent terms cancel
— only the boundary terms contribute.

5. PATH independent.

(6) Example:

$f(\vec{x}) = x^2 y z$

Path-1: $\underset{\vec{a}}{(0,0,0)} \to (1,0,0) \to (1,1,0) \to \underset{\vec{b}}{(1,1,1)}$

$$\vec{\nabla} f = 2xyz\,\hat{i} + x^2 z\,\hat{j} + x^2 y\,\hat{k}$$

$$\int_{\vec{a}}^{\vec{b}} d\vec{l} \cdot \vec{\nabla} f = \int_{(0,0,0)}^{(1,0,0)} dx\,\hat{i} \cdot \{\vec{\nabla} f\}_{\substack{y=0\\z=0}} + \int_{(1,0,0)}^{(1,1,0)} dy\,\hat{j} \cdot \{\vec{\nabla} f\}_{\substack{x=1\\z=0}}$$

$$+ \int_{(1,1,0)}^{(1,1,1)} dz\,\hat{k} \cdot \{\vec{\nabla} f\}_{\substack{x=1\\y=1}}$$

$$= \int_0^1 dx\; 2xyz\Big|_{\substack{y=0\\z=0}} + \int_0^1 dy\; x^2 z\Big|_{\substack{x=1\\z=0}} + \int_0^1 dz\; x^2 y\Big|_{\substack{x=1\\y=1}}$$

$$= 0 + 0 + 1 = 1.$$

$$f(\vec{b}) - f(\vec{a}) = f(1,1,1) - f(0,0,0)$$
$$= 1 - 0 = 1.$$

Path-2: $(0,0,0) \to (1,1,1)$

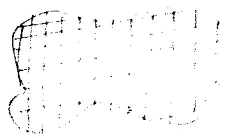
$d\vec{l} = dx\,(\hat{i} + \hat{j} + \hat{k})$ (along the particular path we have $dx = dy = dz$)

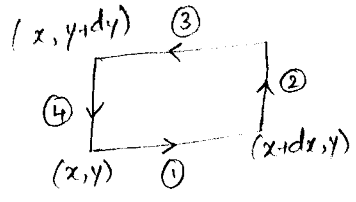
$$\vec{\nabla} f\big|_{x=y=z} = x^3\,(2\hat{i} + \hat{j} + \hat{k})$$

$$\int_{\vec{a}}^{\vec{b}} d\vec{l} \cdot \vec{\nabla} f = \int_0^1 dx\; x^3\, 4 = \frac{x^4}{4} \cdot 4 \Big|_0^1 = 1.$$

III Theorem of curl (Stoke's theorem)

① $$\int_S d\vec{a}\cdot\vec{\nabla}\times\vec{A} = \oint_{\substack{\text{enclosing}\\ \text{curve}}} d\vec{l}\cdot\vec{A}$$





$d\vec{a} = d\vec{l}_1\times d\vec{l}_2$

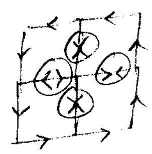
② Consider an infinitesimal rectangle

$$d\vec{a} = \hat{z}\, dx\, dy$$

$$(\vec{\nabla}\times\vec{A})_z = \left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)\hat{z}$$

③ $$\begin{aligned}
d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) &= dx\,dy\left[\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right]\\
&= dy\left[A_y(x+dx,y) - A_y(x,y)\right] - dx\left[A_x(x,y+dy) - A_x(x,y)\right]\\
&= \underbrace{dy\,A_y(x+dx,y)}_{②}\ \underbrace{-\,dy\,A_y(x,y)}_{④}\ \underbrace{-\,dx\,A_x(x,y+dy)}_{③} + \underbrace{dx\,A_x(x,y)}_{①}\\
&= d\vec{l}\cdot\vec{A} \qquad (d\vec{l}\text{ is anticlockwise})
\end{aligned}$$

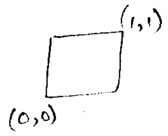
④ – convince that adjacent elements cancel.



– only the boundary survives.

(5) Example :

$$\vec{A} = x^2 y \, \hat{j}$$



Surface : square with (0,0) and (1,1)

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 y & 0 \end{vmatrix} = 2xy \, \hat{k}$$

$$d\vec{a} = \hat{k} \, dx dy$$

$$\int_S d\vec{a} \cdot \vec{\nabla} \times \vec{A} = \int_0^1 \int_0^1 dx dy \, \hat{k} \cdot 2xy \, \hat{k}$$

$$= \int_0^1 dx \, 2x \int_0^1 dy \, y = \frac{1}{2}$$

$$\oint d\vec{l} \cdot \vec{A} = \int_0^1 dx \, \hat{i} \cdot \vec{A}(x,0,0) + \int_0^1 dy \, \hat{j} \cdot \vec{A}(1,y,0)$$

$$+ \int_1^0 dx \, \hat{i} \cdot \vec{A}(x,1,0) + \int_1^0 dy \, \hat{j} \cdot \vec{A}(0,y,0)$$

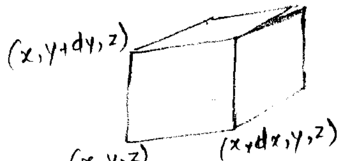
$$= 0 + \int_0^1 dy \, x^2 y \big|_{x=1} + 0 + \int_1^0 dy \, x^2 y \big|_{x=0}$$

$$= \frac{1}{2} .$$

IV Theorem of divergence

① $\int_V d^3x \ \vec{\nabla} \cdot \vec{A} = \oint d\vec{a} \cdot \vec{A}$

② Consider an infinitesimal volume element

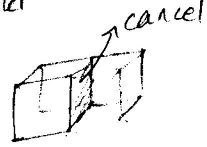


③ $$\begin{aligned} d^3x \ \vec{\nabla} \cdot \vec{A} &= dx\,dy\,dz \left[\frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z \right] \\ &= dy\,dz \left[A_x(x+dx, y, z) - A_x(x, y, z) \right] \\ &\quad + dx\,dz \left[A_y(x, y+dy, z) - A_y(x, y, z) \right] \\ &\quad + dx\,dy \left[A_z(x, y, z+dz) - A_z(x, y, z) \right] \end{aligned}$$

④ – the 6 terms are contributions from 6 faces.

– the negative sign makes sure that the area vector, $d\vec{a}$, is the outward normal (for a closed surface).

– the contributions from adjacent volumes cancel, such that only the surface contributes.



(5) Example :

$\vec{A} = x^2 z \, \hat{k}$.

Volume : cube with corners at $(0,0,0)$ and $(1,1,1)$.

$\vec{\nabla} \cdot \vec{A} = x^2$

$$\int_V d^3x \; \vec{\nabla} \cdot \vec{A} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \; x^2$$

$$= \frac{1}{3}.$$

only 2 out of 6 faces contribute

$$\oint d\vec{a} \cdot \vec{A} = \int_0^1 dx \int_0^1 dy \; \hat{k} \cdot \vec{A}(x,y,1) + \int_0^1 dx \int_0^1 dy \, (-\hat{k}) \cdot \vec{A}(x,y,0)$$

$$= \int_0^1 dx \int_0^1 dy \; x^2 z \big|_{z=1} - \int_0^1 dx \int_0^1 dy \; x^2 z \big|_{z=0}$$

$$= \frac{1}{3} - 0 = \frac{1}{3}.$$