SET - 05

(Cylindrical) Curvilinear coordinates

(1) $x = \rho \cos\phi \qquad \rho = \sqrt{x^2 + y^2}$

$y = \rho \sin\phi \qquad \phi = \tan^{-1}\frac{y}{x}$

$z = z' \qquad z' = z$

(2) $dx = d\rho \frac{\partial}{\partial \rho}x + d\phi \frac{\partial x}{\partial \phi} + dz' \frac{\partial x}{\partial z'}$

$= d\rho \cos\phi - \rho \, d\phi \sin\phi$

$dy = d\rho \sin\phi + \rho \, d\phi \cos\phi$

$dz = dz'$

$\tan\phi = \frac{y}{x}$

$\sec^2\phi \frac{\partial \phi}{\partial x} = -\frac{y}{x^2}$

$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{\rho}$

(3) $\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}$

$= \cos\phi \frac{\partial}{\partial \rho} - \sin\phi \frac{1}{\rho}\frac{\partial}{\partial \phi}$

$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial \rho} + \cos\phi \frac{1}{\rho}\frac{\partial}{\partial \phi}$

$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$

(4) $dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\phi^2 + dz'^2$

$= h_1^2 d\rho^2 + h_2^2 d\phi^2 + h_3^2 dz'^2$

$h_1 = 1$

$h_2 = \rho$

$h_3 = 1$

(5)

$$\hat{\rho} = \frac{\vec{\nabla}\rho}{|\vec{\nabla}\rho|} = \frac{1}{|\vec{\nabla}\rho|}\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\rho$$

$$= \frac{1}{|\vec{\nabla}\rho|}\left[\hat{i}\cos\phi + \hat{j}\sin\phi\right]$$

$$= \hat{i}\cos\phi + \hat{j}\sin\phi$$

$$\hat{\phi} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{1}{|\vec{\nabla}\phi|}\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\phi$$

$$= \frac{1}{|\vec{\nabla}\phi|}\left[-\frac{1}{\rho}\sin\phi\,\hat{i} + \frac{1}{\rho}\cos\phi\,\hat{j} + 0\,\hat{k}\right]$$

$$= -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\hat{z}' = \frac{\vec{\nabla}z'}{|\vec{\nabla}z'|} = \hat{k}$$

(6) Thus, we have.

$$\hat{\rho} = \hat{i}\cos\phi + \hat{j}\sin\phi + 0\,\hat{k}$$

$$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi + 0\,\hat{k}$$

$$\hat{z}' = \hat{i}\,0 + \hat{j}\,0 + \hat{k}$$

(7) Inverting (6) we have

$$\hat{i} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi} + 0\,\hat{z}'$$

$$\hat{j} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi} + 0\,\hat{z}'$$

$$\hat{k} = 0\,\hat{\rho} + 0\,\hat{\phi} + \hat{z}'$$

(8) $$d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$= (\cos\phi\, d\rho - \sin\phi\, \rho\, d\phi)(\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi})$$

$$+ (\sin\phi\, d\rho + \cos\phi\,\rho\, d\phi)(\sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}) + dz'\,\hat{z}'$$

$$= d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz'\,\hat{z}' \rightarrow h_1\, d\rho\,\hat{\rho} + h_2\, d\phi\,\hat{\phi} + h_3\, dz'\,\hat{z}'$$

(9) $$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$= (\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi})\left[\cos\phi\frac{\partial}{\partial\rho} - \sin\phi\,\frac{1}{\rho}\frac{\partial}{\partial\phi}\right]$$

$$+ (\sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi})\left[\sin\phi\frac{\partial}{\partial\rho} + \cos\phi\,\frac{1}{\rho}\frac{\partial}{\partial\phi}\right] + \hat{z}'\frac{\partial}{\partial z'}$$

$$= \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}'\frac{\partial}{\partial z'} \rightarrow \hat{\rho}\frac{1}{h_1}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{h_2}\frac{\partial}{\partial\phi} + \hat{z}'\frac{1}{h_3}\frac{\partial}{\partial z'}$$

(10) Define :

$$f(x,y,z) = f'(\rho,\phi,z')$$

$$\vec{A}(x,y,z) = \hat{\rho}\,A_\rho(\rho,\phi,z') + \hat{\phi}\,A_\phi(\rho,\phi,z') + \hat{z}'\,A_{z'}(\rho,\phi,z')$$

(11) $$\vec{\nabla} f(x,y,z) = \left[\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}'\frac{\partial}{\partial z'}\right] f(x,y,z)$$

$$= \left[\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}'\frac{\partial}{\partial z'}\right] f'(\rho,\phi,z')$$

12 $$\vec{\nabla} \cdot \vec{A} = \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \right] \cdot \left[\hat{\rho} A_\rho + \hat{\phi} A_\phi + \hat{z}' A_{z'} \right]$$

13 $$\begin{bmatrix} \frac{\partial}{\partial \rho} \hat{\rho} & \frac{\partial}{\partial \rho} \hat{\phi} & \frac{\partial}{\partial \rho} \hat{z}' \\ \frac{\partial}{\partial \phi} \hat{\rho} & \frac{\partial}{\partial \phi} \hat{\phi} & \frac{\partial}{\partial \phi} \hat{z}' \\ \frac{\partial}{\partial z'} \hat{\rho} & \frac{\partial}{\partial z'} \hat{\phi} & \frac{\partial}{\partial z'} \hat{z}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \hat{\phi} & -\hat{\rho} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\vec{\nabla} \cdot \vec{A} = \hat{\rho} \cdot \hat{\rho} \frac{\partial}{\partial \rho} A_\rho + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} \cdot \left[\hat{\rho} A_\rho + \hat{\phi} A_\phi \right] + \hat{z}' \cdot \hat{z}' \frac{\partial}{\partial z'} A_{z'}$$

$$= \frac{\partial}{\partial \rho} A_\rho + \hat{\phi} \frac{1}{\rho} \cdot \left(\frac{\partial}{\partial \phi} \hat{\rho} \right) A_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z'} A_{z'}$$

$$= \frac{\partial}{\partial \rho} A_\rho + \frac{1}{\rho} A_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z'} A_{z'}$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z'} A_{z'}$$

$$\rightarrow \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \rho} (h_2 h_3 A_\rho) + \frac{\partial}{\partial \phi} (h_1 h_3 A_\phi) + \frac{\partial}{\partial z'} (h_1 h_2 A_{z'}) \right]$$

(15)
$$\vec{\nabla} \times \vec{A} = \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \right] \times \left[\hat{\rho} A_\rho + \hat{\phi} A_\phi + \hat{z}' A_{z'} \right]$$

$$= 0 + \hat{\rho} \times \hat{\phi} \frac{\partial}{\partial \rho} A_\phi + \hat{\rho} \times \hat{z}' \frac{\partial}{\partial \rho} A_{z'}$$

$$+ \hat{\phi} \frac{1}{\rho} \times \frac{\partial}{\partial \phi} (\hat{\rho} A_\rho) + \hat{\phi} \frac{1}{\rho} \times \frac{\partial}{\partial \phi} (\hat{\phi} A_\phi) + \hat{\phi} \times \hat{z}' \frac{1}{\rho} \frac{\partial}{\partial \phi} A_{z'}$$

$$+ \hat{z}' \times \hat{\rho} \frac{\partial}{\partial z'} A_\rho + \hat{z}' \times \hat{\phi} \frac{\partial}{\partial z'} A_\phi + 0$$

$$= \underset{\checkmark}{\hat{z}'} \frac{\partial}{\partial \rho} A_\phi - \underset{\checkmark}{\hat{\phi}} \frac{\partial}{\partial \rho} A_{z'} - \underset{\checkmark}{\hat{z}'} \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\rho + \underset{\checkmark}{\hat{z}'} \frac{1}{\rho} A_\phi + \underset{\checkmark}{\hat{\rho}} \frac{1}{\rho} \frac{\partial}{\partial \phi} A_{z'}$$

$$+ \underset{\checkmark}{\hat{\phi}} \frac{\partial}{\partial z'} A_\rho - \underset{\checkmark}{\hat{\rho}} \frac{\partial}{\partial z'} A_\phi$$

$$= \hat{\rho} \left[\frac{1}{\rho} \frac{\partial}{\partial \phi} A_{z'} - \frac{\partial}{\partial z'} A_\phi \right] - \hat{\phi} \left[\frac{\partial}{\partial \rho} A_{z'} - \frac{\partial}{\partial z'} A_\rho \right]$$

$$+ \hat{z}' \left[\frac{\partial}{\partial \rho} A_\phi + \frac{1}{\rho} A_\phi - \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\rho \right]$$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z}' \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z'} \\ A_\rho & \rho A_\phi & A_{z'} \end{vmatrix}$$

$$\rightarrow \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\rho} & h_2 \hat{\phi} & h_3 \hat{z}' \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z'} \\ h_1 A_\rho & h_2 A_\phi & h_3 A_{z'} \end{vmatrix}$$

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$$\nabla^2 = \left[\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}'\frac{\partial}{\partial z'}\right]\cdot\left[\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}'\frac{\partial}{\partial z'}\right]$$

$$= \frac{\partial^2}{\partial\rho^2} + \hat{\phi}\frac{1}{\rho}\cdot\frac{\partial}{\partial\phi}\left[\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right] + \frac{\partial^2}{\partial z'^2}$$

$$= \frac{\partial^2}{\partial\rho^2} + \hat{\phi}\frac{1}{\rho}\cdot\left[\hat{\phi}\frac{\partial}{\partial\rho} + \hat{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\rho} - \hat{\rho}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\phi}\frac{\partial}{\partial\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right] + \frac{\partial^2}{\partial z'^2}$$

$$= \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z'^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z'^2}$$

$$\rightarrow \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\rho}\frac{h_2 h_3}{h_1}\frac{\partial}{\partial\rho} + \frac{\partial}{\partial\phi}\frac{h_1 h_3}{h_2}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial z'}\frac{h_1 h_2}{h_3}\frac{\partial}{\partial z'}\right]$$