

(Cylindrical) Curvilinear coordinate

$$\begin{aligned} \textcircled{1} \quad x &= \rho \cos \phi & \rho &= \sqrt{x^2 + y^2} \\ y &= \rho \sin \phi & \phi &= \tan^{-1} \frac{y}{x} \\ z &= z' & z' &= z \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad dx &= d\rho \frac{\partial x}{\partial \rho} + d\phi \frac{\partial x}{\partial \phi} + dz' \frac{\partial x}{\partial z'} \\ &= d\rho \cos \phi - \rho d\phi \sin \phi \\ dy &= d\rho \sin \phi + \rho d\phi \cos \phi \\ dz &= dz' \end{aligned}$$

$$\begin{aligned} \tan \phi &= \frac{y}{x} \\ \sec^2 \phi \frac{\partial \phi}{\partial x} &= -\frac{y}{x^2} \\ \frac{\partial \phi}{\partial x} &= -\frac{\sin \phi}{\rho} \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad \frac{\partial}{\partial x} &= \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} + \frac{\partial z'}{\partial x} \frac{\partial}{\partial z'} \\ &= \cos \phi \frac{\partial}{\partial \rho} - \sin \phi \frac{1}{\rho} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y} &= \sin \phi \frac{\partial}{\partial \rho} + \cos \phi \frac{1}{\rho} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial z'} \end{aligned}$$

$$\begin{aligned} \textcircled{4} \quad dx^2 + dy^2 + dz^2 &= d\rho^2 + \rho^2 d\phi^2 + dz'^2 \\ &= h_1^2 d\rho^2 + h_2^2 d\phi^2 + h_3^2 dz'^2 \end{aligned}$$

$$\begin{aligned} h_1 &= 1 \\ h_2 &= \rho \\ h_3 &= 1 \end{aligned}$$

$$\begin{aligned}
 \hat{s} &= \frac{\vec{\nabla} \phi}{|\vec{\nabla} \phi|} = \frac{1}{|\vec{\nabla} \phi|} \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \phi \\
 &= \frac{1}{|\vec{\nabla} \phi|} \left[\hat{i} \cos \phi + \hat{j} \sin \phi \right] \\
 &= \hat{i} \cos \phi + \hat{j} \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 \hat{\phi} &= \frac{\vec{\nabla} \phi}{|\vec{\nabla} \phi|} = \frac{1}{|\vec{\nabla} \phi|} \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \phi \\
 &= \frac{1}{|\vec{\nabla} \phi|} \left[-\frac{1}{\phi} \sin \phi \hat{i} + \frac{1}{\phi} \cos \phi \hat{j} + 0 \hat{k} \right] \\
 &= -\sin \phi \hat{i} + \cos \phi \hat{j}
 \end{aligned}$$

$$\hat{z}' = \frac{\vec{\nabla} z'}{|\vec{\nabla} z'|} = \hat{k}$$

⑥ Thus, we have.

$$\begin{aligned}
 \hat{s} &= \hat{i} \cos \phi + \hat{j} \sin \phi + 0 \hat{k} \\
 \hat{\phi} &= -\hat{i} \sin \phi + \hat{j} \cos \phi + 0 \hat{k} \\
 \hat{z}' &= \hat{i} 0 + \hat{j} 0 + \hat{k}
 \end{aligned}$$

⑦ Inverting ⑥ we have

$$\begin{aligned}
 \hat{i} &= \cos \phi \hat{s} - \sin \phi \hat{\phi} + 0 \hat{z}' \\
 \hat{j} &= \sin \phi \hat{s} + \cos \phi \hat{\phi} + 0 \hat{z}' \\
 \hat{k} &= 0 \hat{s} + 0 \hat{\phi} + \hat{z}'
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{8} \quad d\vec{l} &= dx \hat{i} + dy \hat{j} + dz \hat{k} \\
 &= (\cos\phi \, d\rho - \sin\phi \, \rho \, d\phi) (\cos\phi \hat{\rho} - \sin\phi \hat{\phi}) \\
 &\quad + (\sin\phi \, d\rho + \cos\phi \, \rho \, d\phi) (\sin\phi \hat{\rho} + \cos\phi \hat{\phi}) + dz' \hat{z}' \\
 &= d\rho \hat{\rho} + \rho \, d\phi \hat{\phi} + dz' \hat{z}' \rightarrow h_1 d\rho \hat{\rho} + h_2 d\phi \hat{\phi} + h_3 dz' \hat{z}'
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{9} \quad \vec{\nabla} &= \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \\
 &= (\cos\phi \hat{\rho} - \sin\phi \hat{\phi}) \left[\cos\phi \frac{\partial}{\partial \rho} - \sin\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} \right] \\
 &\quad + (\sin\phi \hat{\rho} + \cos\phi \hat{\phi}) \left[\sin\phi \frac{\partial}{\partial \rho} + \cos\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} \right] + \hat{z}' \frac{\partial}{\partial z'} \\
 &= \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \rightarrow \hat{\rho} \frac{1}{h_1} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{h_2} \frac{\partial}{\partial \phi} + \hat{z}' \frac{1}{h_3} \frac{\partial}{\partial z'}
 \end{aligned}$$

⑩ Define:

$$f(x, y, z) = f'(\rho, \phi, z')$$

$$\vec{A}(x, y, z) = \hat{\rho} A_\rho(\rho, \phi, z') + \hat{\phi} A_\phi(\rho, \phi, z') + \hat{z}' A_{z'}(\rho, \phi, z')$$

$$\begin{aligned}
 \textcircled{11} \quad \vec{\nabla} f(x, y, z) &= \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \right] f(x, y, z) \\
 &= \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \right] f'(\rho, \phi, z')
 \end{aligned}$$

$$(12) \quad \vec{\nabla} \cdot \vec{A} = \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \right] \cdot \left[\hat{\rho} A_{\rho} + \hat{\phi} A_{\phi} + \hat{z}' A_{z'} \right]$$

$$(13) \quad \begin{bmatrix} \frac{\partial}{\partial \rho} \hat{\rho} & \frac{\partial}{\partial \rho} \hat{\phi} & \frac{\partial}{\partial \rho} \hat{z}' \\ \frac{\partial}{\partial \phi} \hat{\rho} & \frac{\partial}{\partial \phi} \hat{\phi} & \frac{\partial}{\partial \phi} \hat{z}' \\ \frac{\partial}{\partial z'} \hat{\rho} & \frac{\partial}{\partial z'} \hat{\phi} & \frac{\partial}{\partial z'} \hat{z}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \hat{\phi} & -\hat{\rho} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(14) Using (13) in (12)

$$\begin{aligned} \vec{\nabla} \cdot \vec{A} &= \hat{\rho} \cdot \hat{\rho} \frac{\partial}{\partial \rho} A_{\rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} \cdot \left[\hat{\rho} A_{\rho} + \hat{\phi} A_{\phi} \right] + \hat{z}' \cdot \hat{z}' \frac{\partial}{\partial z'} A_{z'} \\ &= \frac{\partial}{\partial \rho} A_{\rho} + \hat{\phi} \frac{1}{\rho} \cdot \left(\frac{\partial}{\partial \phi} \hat{\rho} \right) A_{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_{\phi} + \frac{\partial}{\partial z'} A_{z'} \\ &= \frac{\partial}{\partial \rho} A_{\rho} + \frac{1}{\rho} A_{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_{\phi} + \frac{\partial}{\partial z'} A_{z'} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_{\rho}) + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_{\phi} + \frac{\partial}{\partial z'} A_{z'} \\ &\rightarrow \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \rho} (h_2 h_3 A_{\rho}) + \frac{\partial}{\partial \phi} (h_1 h_3 A_{\phi}) + \frac{\partial}{\partial z'} (h_1 h_2 A_{z'}) \right] \end{aligned}$$

$$\begin{aligned}
(15) \quad \vec{\nabla} \times \vec{A} &= \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \right] \times \left[\hat{\rho} A_\rho + \hat{\phi} A_\phi + \hat{z}' A_{z'} \right] \\
&= 0 + \hat{\rho} \times \hat{\phi} \frac{\partial}{\partial \rho} A_\phi + \hat{\rho} \times \hat{z}' \frac{\partial}{\partial \rho} A_{z'} \\
&\quad + \hat{\phi} \frac{1}{\rho} \times \frac{\partial}{\partial \phi} (\hat{\rho} A_\rho) + \hat{\phi} \frac{1}{\rho} \times \frac{\partial}{\partial \phi} (\hat{\phi} A_\phi) + \hat{\phi} \times \hat{z}' \frac{1}{\rho} \frac{\partial}{\partial \phi} A_{z'} \\
&\quad + \hat{z}' \times \hat{\rho} \frac{\partial}{\partial z'} A_\rho + \hat{z}' \times \hat{\phi} \frac{\partial}{\partial z'} A_\phi + 0 \\
&= \hat{z}' \frac{\partial}{\partial \rho} A_\phi - \hat{\phi} \frac{\partial}{\partial \rho} A_{z'} - \hat{z}' \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\rho + \hat{z}' \frac{1}{\rho} A_\phi + \hat{\rho} \frac{1}{\rho} \frac{\partial}{\partial \phi} A_{z'} \\
&\quad + \hat{\phi} \frac{\partial}{\partial z'} A_\rho - \hat{\rho} \frac{\partial}{\partial z'} A_\phi \\
&= \hat{\rho} \left[\frac{1}{\rho} \frac{\partial}{\partial \phi} A_{z'} - \frac{\partial}{\partial z'} A_\phi \right] - \hat{\phi} \left[\frac{\partial}{\partial \rho} A_{z'} - \frac{\partial}{\partial z'} A_\rho \right] \\
&\quad + \hat{z}' \left[\frac{\partial}{\partial \rho} A_\phi + \frac{1}{\rho} A_\phi - \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\rho \right] \\
&= \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z}' \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z'} \\ A_\rho & \rho A_\phi & A_{z'} \end{vmatrix} \\
&\rightarrow \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\rho} & h_2 \hat{\phi} & h_3 \hat{z}' \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z'} \\ h_1 A_\rho & h_2 A_\phi & h_3 A_{z'} \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
(16) \quad \nabla^2 &= \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \right] \cdot \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z}' \frac{\partial}{\partial z'} \right] \\
&= \frac{\partial^2}{\partial \rho^2} + \hat{\phi} \frac{1}{\rho} \cdot \frac{\partial}{\partial \phi} \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} \right] + \frac{\partial^2}{\partial z'^2} \\
&= \frac{\partial^2}{\partial \rho^2} + \hat{\phi} \frac{1}{\rho} \cdot \left[\hat{\phi} \frac{\partial}{\partial \rho} + \hat{\rho} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \rho} - \hat{\rho} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\phi} \frac{\partial}{\partial \phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} \right] + \frac{\partial^2}{\partial z'^2} \\
&= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z'^2} \\
&= \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z'^2} \\
&\rightarrow \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \rho} \frac{h_2 h_3}{h_1} \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi} \frac{h_1 h_3}{h_2} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial z'} \frac{h_1 h_2}{h_3} \frac{\partial}{\partial z'} \right]
\end{aligned}$$