Delta function

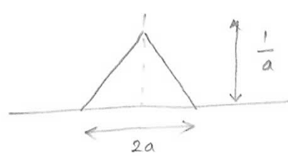
(1) $$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

and satisfies

$$\int_{-\infty}^{+\infty} dx \; f(x)\, \delta(x) = f(0)$$

for a 'well defined' function $f(x)$.

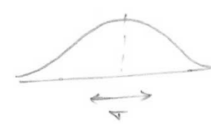
(2) Visual example :



$$\text{Area} = \frac{1}{2}(2a)\frac{1}{a} = 1$$

$$\underset{a \to 0}{Lt}\ (\text{Area}) = 1$$

(3) $$\delta(x) = \underset{\sigma \to 0}{Lt} \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{x^2}{2\sigma}}$$



$$\int_{-\infty}^{+\infty} dx\, \delta(x) = \underset{\sigma \to 0}{Lt} \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{+\infty} dx\; e^{-\frac{x^2}{2\sigma}}$$

$$= \underset{\sigma \to 0}{Lt} \frac{1}{\sqrt{2\pi\sigma}} \sqrt{\pi\, 2\sigma}$$

$$= 1$$

$$\delta(x \neq 0) = 0$$

$$\delta(x = 0) = \infty$$

4 Let
$$\delta(x) = \underset{\sigma\to 0}{Lt} \frac{1}{\pi} \frac{\sigma}{x^2+\sigma^2}$$

$$\delta(0) = \underset{\sigma\to 0}{Lt} \frac{1}{\pi}\frac{1}{\sigma} = \infty$$

$$\delta(x\neq 0) = \underset{\sigma\to 0}{Lt} \frac{1}{\pi}\frac{\sigma}{x^2} = 0$$

$$\int_{-\infty}^{+\infty} dx\ \delta(x) = \underset{\sigma\to 0}{Lt} \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} dx \frac{1}{x^2+\sigma^2}$$

$x = \sigma \tan\theta$

$dx = \sigma \sec^2\theta\, d\theta$.

$$= \underset{\sigma\to 0}{Lt} \frac{\sigma}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sigma \sec^2\theta\, d\theta}{\sigma^2 \sec^2\theta}$$

$$= 1$$

This integral can be done using contour integral too.

5
$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$\underline{a\geq 0}$:
$$\int_{-\infty}^{+\infty} dx\ \delta(ax) f(x) = \int_{-\infty}^{+\infty} \frac{dy}{a}\ \delta(y)\ f\left(\frac{y}{a}\right)$$

$y = ax$

$$= \frac{1}{a} f(0)$$

$\underline{a<0}$:
$$\int_{-\infty}^{+\infty} dx\ \delta(ax) f(x) = \int_{+\infty}^{-\infty} \frac{dy}{a}\ \delta(y)\ f\left(\frac{y}{a}\right)$$

$y = ax$.

$$= -\frac{1}{a} f(0)$$

$$\Rightarrow \quad \delta(ax) = \frac{1}{|a|}\delta(x)$$

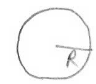
6 A point charge.

$$\rho(x,y,z) = q\,\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

$$\int d^3x\; \rho(x,y,z) = q.$$

7 A circular loop of charge.

total charge = Q.



let $\rho(\rho,\phi,z) = c\,\delta(z-0)\,\delta(\rho-R)$

$$\int_0^{\infty}\rho\, d\rho \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dz = \int_0^{\infty} \rho\, d\rho \; 2\pi \int_{-\infty}^{+\infty} dz \; c\,\delta(z)\,\delta(\rho-R)$$

$$= 2\pi c R = Q$$

$$\Rightarrow \quad c = \frac{Q}{2\pi R}$$

Thus, $\rho(\rho,\phi,z) = \dfrac{Q}{2\pi R}\,\delta(z)\,\delta(\rho-R)$

8 Remember

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^3} = \frac{3}{r^3} + \vec{r} \cdot \frac{(-3)}{r^4} \hat{r} = 0$$

Thus, $$\int d^3r \; \vec{\nabla} \cdot \frac{\vec{r}}{r^3} \stackrel{?}{=} 0$$

Consider

$$\int_V d^3r \; \vec{\nabla} \cdot \frac{\vec{r}}{r^3} = \oint d\vec{a} \cdot \frac{\vec{r}}{r^3}$$

$$= 4\pi R^2 \; \hat{R} \cdot \frac{\vec{R}}{R^3}$$

$$= 4\pi$$

Conclude that

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^3} = 4\pi \; \delta^{(3)}(\vec{r}) .$$