SET-07

Maxwell's equations in SI units

(1) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$-\vec{\nabla}\times\vec{E} = \frac{\partial \vec{B}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0\vec{J}$

and the corresponding force on a test charge q is given by the Lorentz force

$$\vec{F} = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

$\mu_0 = 4\pi\times10^{-7}\ \frac{\text{Tesla-meter}}{\text{Ampere}}$.

$c = 299\,729\,458$ m/s (EXACT!)

$\epsilon_0\mu_0 = \frac{1}{c^2}$

(2) Define

$$\vec{D} = \epsilon_0\vec{E}$$
$$\vec{B} = \mu_0\vec{H}$$

in terms of which we have

$\vec{\nabla}\cdot\vec{D} = \rho$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$-\vec{\nabla}\times\vec{E} = \frac{\partial \vec{B}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$.

(3) $$\boxed{\rho(\vec{x},t)} \longrightarrow \boxed{\vec{E}(\vec{x},t),\ \vec{B}(\vec{x},t)} \longrightarrow \boxed{\text{test charge } q}$$

→ back reaction will be neglected in this course. Complete study requires quantum electrodynamics.

→ It is crucial to realize that $\vec{E}$ and $\vec{B}$, though distributed over space and thus unlocalized, have energy and momentum associated with. Eg: A laser beam exerts pressure on a mirror.

Statement of conservation of charge

(4) The two inhomogeneous Maxwell's equations are

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{--- (i)}$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J} \quad \text{--- (ii)}$$

(5) Using (4) - (ii) we have, taking divergence,

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{H} = \vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{\nabla}\cdot\vec{J}$$

(6)
$$\begin{aligned}
\vec{\nabla}\cdot\vec{\nabla}\times\vec{H} &= \epsilon_{ijk}\,\nabla_i\nabla_j H_k \\
&= \tfrac{1}{2}\,\epsilon_{ijk}\,(\nabla_i\nabla_j + \nabla_j\nabla_i)\,H_k \qquad \left(\text{using } \frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}, \text{ etc.}\right) \\
&= \tfrac{1}{2}\,(\epsilon_{ijk}\nabla_i\nabla_j + \epsilon_{\underset{j}{i}\,\underset{i}{j}\,k}\,\nabla_{\underset{i}{j}}\nabla_{\underset{j}{i}})\,H_k \qquad (\text{relabelling dummy variables}) \\
&= \tfrac{1}{2}\,(\epsilon_{ijk}\nabla_i\nabla_j + \epsilon_{jik}\nabla_i\nabla_j)\,H_k \\
&= \tfrac{1}{2}\,(\epsilon_{ijk} + \epsilon_{jik})\,\nabla_i\nabla_j H_k \qquad (\because \epsilon_{ijk} = -\epsilon_{jik}) \\
&= \tfrac{1}{2}\,(\epsilon_{ijk} - \epsilon_{ijk})\,\nabla_i\nabla_j H_k \\
&= 0
\end{aligned}$$

(7) Note, the above proof uses the fact that a symmetric combination $(\nabla_i\nabla_j)$ is not compatible with an anti-symmetric combination (ϵ_{ijk}). The following could be carelessly judged to be zero:

(i) $(\vec{\nabla}f)\times(\vec{\nabla}g) \neq 0$

(ii) $\vec{A}\cdot(\vec{\nabla}\times\vec{A}) \neq 0$

8 Using 6 in 5 we have

$$\vec{\nabla}\cdot\frac{\partial}{\partial t}\vec{D} + \vec{\nabla}\cdot\vec{j} = 0$$

$$\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{D} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \left(\text{using } \frac{\partial}{\partial x}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}\text{, etc.}\right)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \left(\text{using 4-(i), Maxwell's equation.}\right).$$

This is the statement of charge conservation, which is valid at every point in space.

9 In fluid dynamics the continuity equation relates the mass density and velocity of fluid

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\,\vec{u}) = 0$$

velocity of fluid at a spacial point.

mass density of fluid

10 The analogy with fluid dynamics leads to the interpretation in 8

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

$\rho \rightarrow$ charge density

$\vec{j} \rightarrow$ flux of charge density.

This leads to the interpretation

$$\vec{j} = \rho\,\vec{v}$$

where $\vec{v}$ is the velocity of individual charges in the charge distribution.

11 Integrating the charge conservation equation in 8

$$\int d^3r \, \frac{\partial \rho}{\partial t} + \int_V d^3r \, \vec{\nabla} \cdot \vec{j} = 0$$

$$\frac{\partial}{\partial t} \underbrace{\int d^3r \, \rho}_{Q(t)} + \oint_S d\vec{a} \cdot \vec{j} = 0$$

$$\underbrace{\frac{\partial}{\partial t} Q(t)}_{\text{rate of change of charge inside volume } V.} + \underbrace{\oint_S d\vec{a} \cdot \vec{j}}_{\text{rate of charge flow out of closed surface } S.} = 0$$

This is the statement of conservation of charge in integral form.

Electric and magnetic potential

(12) Let us next look into the information content in the homogeneous Maxwell's equations

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{— (i)}$$

$$\vec{\nabla} \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0 \quad \text{— (ii)}$$

Since these equations donot have ρ and $\vec{J}$ in them they should be interpreted as constraints on the electric and magnetic fields.

(13) $\vec{\nabla} \cdot \vec{B} = 0$ a sufficient condition for the solution is implies that

$\vec{B} = \vec{\nabla} \times A$

because $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$, unless $\vec{A}$ has discontinuities.

(14) Is $\vec{A}(\vec{x}, t)$ unique? No.

15 Using $\vec{B} = \vec{\nabla} \times \vec{A}$ in the other homogeneous (sourceless) Maxwell's equation we have

$$-\vec{\nabla} \times \vec{E} - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = 0$$

$$-\vec{\nabla} \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t} \right] = 0$$

16 Again, since $\vec{\nabla} \times \vec{\nabla}\phi = 0$ we conclude

$$-\left(\vec{E} + \frac{\partial A}{\partial t} \right) = \vec{\nabla}\phi$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

17 → The negative sign in 16 is a conventional choice.

→ Is $\phi(\vec{x}, t)$ unique? No.

18 Thus, we have using 13 and 16

$\vec{A}(\vec{x},t)$ – magnetic vector potential

$\phi(\vec{x},t)$ – electric scalar potential

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

19 The non-uniqueness of ϕ and $\vec{A}$ is called gauge freedom. A gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda(\vec{x}, t)$$

$$\phi' = \phi + \frac{\partial}{\partial t}\lambda(\vec{x}, t)$$

leaves $\vec{E}$ and $\vec{B}$ invariant. Verify this by showing that $\vec{B}' = \vec{\nabla} \times A' = \vec{B}$ and $\vec{E}' = -\vec{\nabla}\phi' - \frac{\partial A'}{\partial t} = \vec{E}$.