SET-08

Electrostatics and Magnetostatics

(1) The Maxwell equations are

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{D} = \epsilon_0\vec{E}$$

$$-\vec{\nabla}\times\vec{E} = \frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J} \qquad \vec{B} = \mu_0\vec{H}$$

and the Lorentz force on a test charge is

$$\vec{F} = q[\vec{E} + \vec{v}\times\vec{B}].$$

(2) Statics is defined by

$$\frac{\partial\rho}{\partial t} = 0, \quad \frac{\partial\vec{J}}{\partial t} = 0, \quad \frac{\partial\vec{E}}{\partial t} = 0, \quad \frac{\partial\vec{H}}{\partial t} = 0.$$

For consistency we require the statement of conservation of charge to be

$$\underbrace{\cancel{\frac{\partial\rho}{\partial t}}}_{=0} + \vec{\nabla}\cdot\vec{J} = 0$$

(3) Thus, we have

Electrostatics

$$\begin{cases} \vec{\nabla}\cdot\vec{D} = \rho \\ \vec{\nabla}\times\vec{E} = 0 \end{cases}$$

$$\begin{cases} \vec{D} = \epsilon_0\vec{E} \\ \vec{E} = -\vec{\nabla}\phi \\ \vec{F} = q\vec{E} \end{cases}$$

Magnetostatics

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{H} = \vec{J} \end{cases} \qquad (\Rightarrow \vec{\nabla}\cdot\vec{J} = 0)$$

$$\begin{cases} \vec{B} = \mu_0\vec{H} \\ \vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{F} = q\,\vec{v}\times\vec{B} \end{cases}$$

Note, $\vec{E}$ and $\vec{B}$ have decoupled.

Uniqueness of solutions in electrostatics

(4) Maxwell's equations for electrostatics are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E} = 0 \qquad (\Rightarrow \ \vec{E} = -\vec{\nabla}\phi)$$

(5) Together, they read

$$-\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

(6) What about boundary conditions? We shall now show that $\vec{E} = 0$ at $r \to \infty$ is a sufficient condition to require unique solutions for $\vec{E}$.

(7) We shall prove this by contradiction. Let $\vec{E}_1$ and $\vec{E}_2$ be distinct (thus non-unique) solutions,

$$\vec{\nabla} \cdot \vec{E}_1 = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{E}_2 = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E}_1 = 0 \qquad \vec{\nabla} \times \vec{E}_2 = 0$$

(8) Subtracting the equations in (7) we have.

$$\vec{\nabla} \cdot (\vec{E}_1 - \vec{E}_2) = 0$$

$$\vec{\nabla} \times (\vec{E}_1 - \vec{E}_2) = 0$$

(13) Divide out $4\pi R^2$. It can be shown (see Schwinger et. al., chapter 1) that $R=0$ does not lead to issues.

$$- \frac{d}{dR}\left(\frac{1}{2}\,\mathcal{E}_x^2\right) + \frac{1}{4\pi R^2}\left(\vec{\nabla}\,\mathcal{E}_x\right)^2 = 0$$

(14) Integrate over R,

$$- \int_0^\infty dR\, \frac{d}{dR}\left(\frac{1}{2}\,\mathcal{E}_x^2\right) + \int_0^\infty dR\, \frac{1}{4\pi R^2}\left(\vec{\nabla}\,\mathcal{E}_x\right)^2 = 0$$

$$- \frac{1}{2}\,\mathcal{E}_x^2\Big|_{\text{at } R=\infty} + \underbrace{\frac{1}{2}\,\mathcal{E}_x^2\Big|_{\text{at } R=0}}_{+} + \underbrace{\int_0^\infty dR\, \frac{1}{4\pi R^2}\left(\vec{\nabla}\,\mathcal{E}_x\right)^2}_{+} = 0$$

(15) Introducing the boundary condition, $\mathcal{E}_x = 0$ at $R \to \infty$, then the two positive terms can sum to zero only if each term is zero. Thus,

(i) $\mathcal{E}_x = 0$, at $R=0$.

(ii) $\vec{\nabla}\,\mathcal{E}_x = 0$, everywhere.

(16) Since, our choice of origin of sphere in (12) was arbitrary we further have

$$\mathcal{E}_x = 0 \quad , \quad \text{everywhere.}$$

(17) Thus, we have contradicted our first assumption

$$\vec{\mathcal{E}} = \vec{E}_1 - \vec{E}_2 = 0 \, ,$$

which implies that the solutions to electrostatics is unique.

Earnshaw's theorem

(18) We ask if an external test charge q can remain stable under the force exerted on it due to electric fields created by charge distribution ρ.

$$\rho \xrightarrow{\text{Maxwell's eqn.}} \vec{E} \xrightarrow{\text{Lorentz force}} q$$

ρ: source of electric field $\vec{E}$.

q: the test charge on which the electric field exerts the force.

(19) Sources ρ create the electric field,

$$\vec{E} = -\vec{\nabla}\phi$$

$$-\nabla^2\phi = \frac{\rho}{\epsilon_0}.$$

The electric field exerts a force on the test charge,

$$\vec{F} = q\vec{E}$$

$$= -q\vec{\nabla}\phi$$

(20) Since the test charge cannot occupy the same position as the sources, we have the electric field at the position of test charge satisfy

$$-\nabla^2\phi = 0$$

(because $\rho = 0$ at the position of test charge.)

(24) We also have

$$\vec{\nabla} \times \underbrace{(\vec{\nabla} \times \vec{E})}_{=0} = \vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{E}}_{=0}) - \nabla^2 \vec{E}$$

$$\Rightarrow \quad \nabla^2 \vec{E} = 0$$

$$\Rightarrow \quad \nabla^2 \vec{\nabla} \phi = 0$$

(25) Thus, we have

$$\vec{\nabla} \phi = 0 , \quad \nabla^2 \phi = 0 , \quad \text{and} \quad \vec{\nabla} \nabla^2 \phi = 0 .$$

And Earnshaw theorem says that it is impossible to have such a ϕ.

(26) In case the stability requirement was stated as.

$$\frac{\partial^2}{\partial x^2} \phi > 0 , \quad \frac{\partial^2 \phi}{\partial y^2} > 0 , \quad \frac{\partial^2}{\partial z^2} > 0 ,$$

which would imply

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi > 0 ,$$

and thus contradict

$$\nabla^2 \phi = 0 .$$

(27) But, as correctly pointed out by class

$$\frac{\partial^2}{\partial x^2}\phi = 0$$

could also lead to stable points. Eg. $\phi = x^4$.

Thus, one has to be careful.

(28) As of today, I am unable to complete the proof for Earnshaw's theorem.

(29) Examples that seemingly contradict Earnshaw's theorem:

(i) Levitron - spinning top (not static)

(ii) Levitating frog (diamagnetism)

(iii) Levitating magnets near surfaces (presence of boundaries)