SET- 09

Point charge

1. The charge density for a point charge is

$$\rho(\vec{r}) = q\, \delta^{(3)}(\vec{r} - \vec{r}_a)$$

$\vec{r}$ – space variable
$\vec{r}_a$ – position of charge.

2. From (electrostatic) Maxwell's equations we have

$$-\nabla^2 \phi(\vec{r}) = \frac{q}{\epsilon_0}\, \delta^{(3)}(\vec{r} - \vec{r}_a)$$

$\rightarrow e^{ikx}$ forms a complete set.

3. Fourier transformation in 1-D

$$f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx}\, \tilde{f}(k)$$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} dx\, e^{-ikx}\, f(x)$$

$$\delta(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx} \cdot 1$$

$$1 = \int_{-\infty}^{+\infty} dx\, e^{-ikx}\, \delta(x)$$

4. Using Fourier transformation in 3-D we can write

$$\phi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r} - \vec{r}_a)}\, \tilde{\phi}(\vec{k})$$

5. Note that

$$\vec{\nabla}\phi = \int \frac{d^3k}{(2\pi)^3}\, \vec{\nabla}\, e^{i\vec{k}\cdot(\vec{r} - \vec{r}_a)}\, \tilde{\phi}(\vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3}\, i\vec{k}\, e^{i\vec{k}\cdot(\vec{r} - \vec{r}_a)}\, \tilde{\phi}(\vec{k})$$

(6) Further,

$$\nabla^2 \phi = \vec{\nabla} \cdot \vec{\nabla} \phi$$

$$= \int \frac{d^3k}{(2\pi)^2} \; i\vec{k} \cdot \vec{\nabla} \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)} \, \tilde{\phi}(\vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3} \; i\vec{k} \cdot i\vec{k} \; e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)} \, \tilde{\phi}(\vec{k})$$

$$= - \int \frac{d^3k}{(2\pi)^3} \; k^2 \; e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)} \, \tilde{\phi}(\vec{k})$$

(7) Also,

$$\delta^{(3)}(\vec{r}-\vec{r}_a) = \int \frac{d^3k}{(2\pi)^3} \; e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)}$$

(8) Using (6) and (7) in (2) we have

$$-\nabla^2 \phi = \frac{q}{\epsilon_0} \, \delta^{(3)}(\vec{r}-\vec{r}_a)$$

$$+ \int \frac{d^3k}{(2\pi)^3} \; k^2 \; e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)} \, \tilde{\phi}(\vec{k}) = \frac{q}{\epsilon_0} \int \frac{d^3k}{(2\pi)^3} \; e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)}$$

which can be written as

$$\int \frac{d^3k}{(2\pi)^3} \; e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)} \left[k^2 \tilde{\phi}(\vec{k}) - \frac{q}{\epsilon_0} \right] = 0$$

(9) Since $e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)}$ form a complete basis set

$$\int \frac{d^3k}{(2\pi)^3} \; e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)} \, \tilde{F}(\vec{k}) = 0$$

$$\Rightarrow \quad \tilde{F}(\vec{k}) = 0$$

Analogy:

$x \hat{i} + y \hat{j} = 0$

$\Rightarrow x = 0, \; y = 0.$

(10) Using (9) in (8)

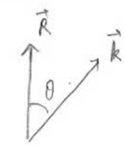
$$k^2\, \tilde{\phi}(\vec{k}) - \frac{q}{\epsilon_0} = 0$$

$$\tilde{\phi}(\vec{k}) = \frac{q}{\epsilon_0}\, \frac{1}{k^2}.$$

(11) Using (10) in (4)

$\vec{r} - \vec{r}_a = \vec{R}$

— choose $\vec{R}$ along the z-direction



$k_x = k \sin\theta \cos\phi$
$k_y = k \sin\theta \sin\phi$
$k_z = k \cos\theta$

$$\phi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)}\, \tilde{\phi}(\vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}_a)}\, \frac{q}{\epsilon_0}\, \frac{1}{k^2}$$

$$= \frac{1}{(2\pi)^3}\, \frac{q}{\epsilon_0} \int_0^\infty k^2\, dk \int_0^\pi \sin\theta\, d\theta\, \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\, e^{ikR\cos\theta}\, \frac{1}{k^2}$$

$$= \frac{1}{(2\pi)^3}\, \frac{q}{\epsilon_0}\, 2\pi \int_0^\infty dk \int_0^\pi \sin\theta\, d\theta\, e^{ikR\cos\theta}$$

$\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$$= \frac{q}{4\pi\epsilon_0}\, \frac{1}{\pi} \int_0^\infty dk \int_{+1}^{-1} (-dt)\, e^{ikRt}$$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$$= \frac{q}{4\pi\epsilon_0}\, \frac{1}{\pi} \int_0^\infty dk\, \frac{1}{ikR} \left(e^{ikR} - e^{-ikR}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\, \frac{2}{\pi} \int_0^\infty \frac{dk}{kR}\, \sin kR$$

$kR = x.$

$\int_0^\infty \frac{dx}{x} \sin x = \frac{\pi}{2}$

$$= \frac{q}{4\pi\epsilon_0}\, \frac{1}{R}\, \frac{2}{\pi} \int_0^\infty \frac{dx}{x}\, \sin x$$

$$= \frac{q}{4\pi\epsilon_0}\, \frac{1}{R}$$

$$= \frac{q}{4\pi\epsilon_0}\, \frac{1}{|\vec{r} - \vec{r}_a|}$$

(12) Thus, we have shown that

$$-\nabla^2 \phi = \frac{q}{\epsilon_0} \delta^{(3)}(\vec{r} - \vec{r}_a)$$

has the solution

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_a|}$$

(13) The electric field due to a point charge is

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi$$

$$= -\vec{\nabla}\left(\frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_a|}\right)$$

$$= -\frac{q}{4\pi\epsilon_0} \vec{\nabla} \frac{1}{|\vec{r} - \vec{r}_a|}$$

$$= -\frac{q}{4\pi\epsilon_0} \vec{\nabla} \frac{1}{\sqrt{(x-x_a)^2 + (y-y_a)^2 + (z-z_a)^2}}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{(x-x_a)\hat{i} + (y-y_a)\hat{j} + (z-z_a)\hat{k}}{\left[(x-x_a)^2 + (y-y_a)^2 + (z-z_a)^2\right]^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_a}{|\vec{r} - \vec{r}_a|^3}$$

(14) Force on a charge, q, at position $\vec{r}_b$ is

$$\vec{F} = q_b \vec{E}(\vec{r}_b) = q_b \frac{q_a}{4\pi\epsilon_0} \frac{\vec{r}_b - \vec{r}_a}{|\vec{r}_b - \vec{r}_a|^3}$$

which is the Coulomb's law.

Green's function

(15) We have seen that the solution to Poisson's equation for a point charge

$$-\nabla^2 \phi = \frac{q}{\epsilon_0} \delta^{(3)}(\vec{r} - \vec{r}_a)$$

is

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_a|}$$

(16) Let us define the Green's function

$$G(\vec{r}, \vec{r}\,') = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}\,'|} \quad \Rightarrow \quad -\nabla^2 G = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

(17) The statement of superposition in electrostatics is

$$-\nabla^2 \phi = \frac{q_1}{\epsilon_0} \delta^{(3)}(\vec{r} - \vec{r}_1) + \frac{q_2}{\epsilon_0} \delta^{(3)}(\vec{r} - \vec{r}_2) + \cdots$$

$$= \sum_i \frac{q_i}{\epsilon_0} \delta^{(3)}(\vec{r} - \vec{r}_i)$$

has solution

$$\phi(\vec{r}) = \frac{q_1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_1|} + \frac{q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_2|} + \cdots$$

$$= \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_i|}$$

(18) For a continuous charge distribution we have.

$$\sum_i q_i \, \delta^{(3)}(\vec{r} - \vec{r}_i) \longrightarrow \int d^3r_i \; \rho(\vec{r}_i) \; \delta^{(3)}(\vec{r} - \vec{r}_i)$$

$$= \rho(\vec{r})$$

(19) Now, consider the Green's function in (16)

$$-\nabla^2 \, G(\vec{r}, \vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

$$-\nabla^2 \, G(\vec{r}, \vec{r}\,') \; \rho(\vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,') \; \rho(\vec{r}\,')$$

$$-\nabla^2 \int d^3r' \; G(\vec{r}, \vec{r}\,') \, \rho(\vec{r}\,') = \int d^3r' \; \delta^{(3)}(\vec{r} - \vec{r}\,') \; \rho(\vec{r}\,')$$

$$-\nabla^2 \int d^3r' \; G(\vec{r}, \vec{r}\,') \; \rho(\vec{r}\,') = \rho(\vec{r})$$

$$-\nabla^2 \left[\frac{1}{\epsilon_0} \int d^3r' \; G(\vec{r}, \vec{r}\,') \; \rho(\vec{r}\,') \right] = \frac{1}{\epsilon_0} \rho(\vec{r})$$

(20) Poisson's equation for a continuous charge distribut is

$$-\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

21 Comparing 19 and 20 we have

$$\phi(\vec{r}) = \frac{1}{\epsilon_0} \int d^3r' \; G(\vec{r}, \vec{r}') \; \rho(\vec{r}')$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \; \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \qquad (\text{using } 18)$$

22 Electric field for a continuous charge distributn will be

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi$$

$$= -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \int d^3r' \; \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= -\frac{1}{4\pi\epsilon_0} \int d^3r' \; \rho(\vec{r}') \; \vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|}$$

direction of $\vec{r}'$ is position dep. So, careful with integration.

$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \; \rho(\vec{r}') \; \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

23 Force on a test charge q_b at position $\vec{r}_b$.

$$\vec{F} = q_b \, \vec{E}(\vec{r}_b)$$

$$= \frac{q_b}{4\pi\epsilon_0} \int d^3r' \; \rho(\vec{r}') \; \frac{\vec{r}_b - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

Dipole – two point charges

(24) We have seen that

$$-\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

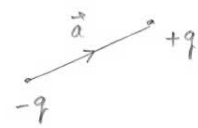
has the solution

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

(25) A dipole consists of two point charges of equal and opposite charge separated by distance a.

The dipole moment is defined as

$$\vec{d} = q\,\vec{a}$$



(26) The charge density for a dipole is

$$\rho(\vec{r}) = q\,\delta^{(3)}\left(\vec{r} - \frac{\vec{a}}{2}\right) - q\,\delta^{(3)}\left(\vec{r} + \frac{\vec{a}}{2}\right)$$

→ origin chosen at the center of dipole.

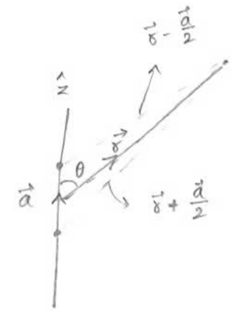
(27) Using (26) in (24)

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{1}{\vec{r} - \vec{r}\,'} \left[q\,\delta^{(3)}\left(\vec{r}\,' - \frac{\vec{a}}{2}\right) - q\,\delta^{(3)}\left(\vec{r}\,' + \frac{\vec{a}}{2}\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{\left|\vec{r} - \frac{\vec{a}}{2}\right|} - \frac{q}{4\pi\epsilon_0} \frac{1}{\left|\vec{r} + \frac{\vec{a}}{2}\right|}$$

(28) Let us consider the approximation

$$|\vec{a}| << |\vec{r}|$$



(29)
$$\frac{1}{\left|\vec{r} \pm \frac{\vec{a}}{2}\right|} = \frac{1}{\sqrt{r^2 + \frac{a^2}{4} \pm \vec{r}\cdot\vec{a}}}$$

$$= \frac{1}{\sqrt{r^2 + \frac{a^2}{4} \pm r a \cos\theta}}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + O(x^2)$$

$$= \frac{1}{r} \frac{1}{\sqrt{1 \pm \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}}}$$

$$= \frac{1}{r}\left[1 \mp \frac{1}{2}\frac{a}{r}\cos\theta + O\left(\frac{a}{r}\right)^2\right]$$

(30) Using (29) in (27)

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left[1 + \frac{1}{2}\frac{a}{r}\cos\theta + O\left(\frac{a}{r}\right)^2\right] - \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left[1 - \frac{1}{2}\frac{a}{r}\cos\theta + O\left(\frac{a}{r}\right)^2\right]$$

$$= \frac{q\, a \cos\theta}{4\pi\epsilon_0 r^2} + O\left(\frac{1}{r}\right)^3$$

$$= \frac{1}{4\pi\epsilon_0}\frac{(\vec{d}\cdot\hat{r})}{r^2} + O\left(\frac{1}{r}\right)^3$$

(31) A point dipole :

$$\vec{d} = q\,\vec{a}$$

$a \to 0$, $q \to \infty$ with fixed d.

Thus, for a point dipole all higher order terms in (30) will be zero because they will all have a single power of q but more than one power of a. Thus, for a point dipole we exactly have

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{(\vec{d}\cdot\hat{r})}{r^2}$$

(32) Electric field due to a point dipole is

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi$$

$$= -\frac{1}{4\pi\epsilon_0}\,\vec{\nabla}\,\frac{(\vec{d}\cdot\hat{r})}{r^2} = -\frac{1}{4\pi\epsilon_0}\,\vec{\nabla}\,\frac{(\vec{d}\cdot\vec{r})}{r^3}$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\left(\vec{\nabla}\frac{1}{r^3}\right)(\vec{d}\cdot\vec{r}) + \frac{1}{r^3}\,\vec{\nabla}(\vec{d}\cdot\vec{r})\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\left(\vec{\nabla}\frac{1}{r^3}\right)(\vec{d}\cdot\vec{r}) + \frac{1}{r^3}\,\vec{d}\right]$$

$$\vec{\nabla}(\vec{d}\cdot\vec{r}) = (\vec{\nabla}\,\vec{r})\cdot\vec{d} = \overleftrightarrow{1}\cdot\vec{d} = \vec{d}$$

$$= -\frac{1}{4\pi\epsilon_0}\left[-\frac{3}{r^4}(\vec{\nabla}r)(\vec{d}\cdot\vec{r}) + \frac{1}{r^3}\,\vec{d}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[-3\,\hat{r}\,(\vec{d}\cdot\hat{r}) + \vec{d}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[3\,\hat{r}\,(\vec{d}\cdot\hat{r}) - \vec{d}\right]$$

(33) If $\vec{d} = d\,\hat{z}$ we have

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}$$

(34) Using

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

we have

$\cos\theta = \frac{z}{r}$

$r^2 = x^2 + y^2 + z^2$

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}\phi \\
&= -\frac{d}{4\pi\epsilon_0}\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\frac{\cos\theta}{r^2} \\
&= -\frac{d}{4\pi\epsilon_0}\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\frac{z}{r^3} \\
&= -\frac{d}{4\pi\epsilon_0}\left[-3\frac{xz}{r^5}\hat{i} - 3\frac{yz}{r^5}\hat{j} + \left(\frac{1}{r^3} - 3\frac{z^2}{r^5}\right)\hat{k}\right] \\
&= \frac{d}{4\pi\epsilon_0}\left[3\frac{z}{r^4}\hat{r} - \frac{1}{r^3}\hat{k}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3d\cos\theta\,\hat{r} - \vec{d}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{d}\cdot\hat{r})\,\hat{r} - \vec{d}\right]
\end{aligned}$$

$\vec{d}\cdot\hat{r} = d\cos\theta$

which verifies (32).

(35) We can also use

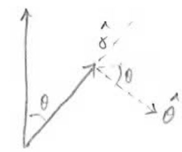
$$\vec{\nabla} = \hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\vec{E} = -\vec{\nabla}\phi$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right]\frac{d\cos\theta}{r^2}$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\hat{r}\,\frac{(-2)\,d\cos\theta}{r^3} + \hat{\theta}\,\frac{1}{r}\,\frac{d}{r^2}(-\sin\theta) + \hat{\phi}\,.\,0\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[+2d\cos\theta\,\hat{r} + d\sin\theta\,\hat{\theta}\right]$$



(36)
$$\vec{d} = (\vec{d}\cdot\hat{r})\,\hat{r} + (\vec{d}\cdot\hat{\theta})\,\hat{\theta}$$
$$= d\cos\theta\,\hat{r} - d\sin\theta\,\hat{\theta}$$

$$\vec{d}\cdot\hat{\theta} = -d\sin\theta$$

(37) Using (36) in (35)

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[2d\cos\theta\,\hat{r} + d\cos\theta\,\hat{r} - \vec{d}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[3(\vec{d}\cdot\hat{r})\,\hat{r} - \vec{d}\right]$$

which again verifies (32).

(38)

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{d}\cdot\hat{r})\,\hat{r} - \vec{d}\right]$$

$\underline{\theta = 0}$

$$\vec{d}\cdot\hat{r} = d\cos 0 = d$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3d\hat{z} - d\hat{z}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2\vec{d}}{r^3}$$

$\underline{\theta = \frac{\pi}{2}}$

$$\vec{d}\cdot\hat{r} = d\cos\frac{\pi}{2} = 0$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[0 - \vec{d}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{\vec{d}}{r^3}$$

$\underline{\theta = \pi}$

$$\vec{d}\cdot\hat{r} = d\cos\pi = -d$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(-d)(-\hat{z}) - d\hat{z}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2\vec{d}}{r^3}$$