SET-10

I Volume charge

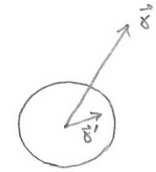
(1) Consider a uniformly charged solid sphere of radius R whose charge density is described by

$$\rho(\vec{r}) = \rho_0 f(r) \qquad f(r) = \begin{cases} 1 & r < R \\ 0 & r > R. \end{cases}$$

let total charge be Q.

(2) The electric potential at point $\vec{r}$ is given by

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$



where

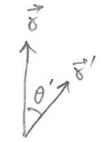
$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2\,\vec{r}\cdot\vec{r}'}$$

$\vec{r} \rightarrow$ observation point

$\vec{r}' \rightarrow$ source point.

(3)
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \int_0^{\infty} r'^2 dr' \, \frac{\rho_0 f(r')}{\sqrt{r^2 + r'^2 - 2\,\vec{r}\cdot\vec{r}'}}$$

(4) For each observation point $\vec{r}$, without any loss of generality, we can choose $\vec{r}$ to be along $\hat{z}$ in the space of $\vec{r}'$.



$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \int_0^{\infty} r'^2 dr' \, \frac{\rho_0 f(r')}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}$$

(5)
$$\phi(\vec{r}) = \frac{\rho_0}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \int_0^{R} r'^2 dr' \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$
(using definition of $f(r)$ in (1))

$$= \frac{\rho_0\, 2\pi}{4\pi\epsilon_0}\int_0^{\pi} \sin\theta' d\theta' \int_0^{R} r'^2 dr' \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$
(completing ϕ integral.)

$$\cos\theta' = t', \qquad -\sin\theta' d\theta' = dt', \qquad \begin{matrix}\theta'=0,\ t'=1\\ \theta'=\pi,\ t'=-1\end{matrix}$$

$$= \frac{\rho_0}{2\epsilon_0}\int_0^{R} r'^2 dr' \int_{-1}^{1} dt' \frac{1}{\sqrt{r^2+r'^2-2rr't'}}$$

$$r^2+r'^2-2rr't' = y' \qquad t'=-1 \Rightarrow y'=(r+r')^2$$
$$-2rr'\,dt' = dy' \qquad t'=1 \Rightarrow y'=(r-r')^2$$

$$= \frac{\rho_0}{2\epsilon_0}\int_0^{R} r'^2 dr' \int_{(r+r')^2}^{(r-r')^2} \frac{dy'}{-2rr'} \frac{1}{\sqrt{y'}}$$

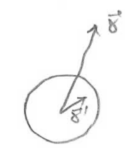
$$= \frac{\rho_0}{4\epsilon_0}\frac{1}{r}\int_0^{R} r' dr' \int_{(r-r')^2}^{(r+r')^2} \frac{dy'}{\sqrt{y'}} \qquad \frac{(y')^{-\frac{1}{2}+1}}{(-\frac{1}{2}+1)} = 2\sqrt{y'}$$

$$= \frac{\rho_0}{2\epsilon_0}\frac{1}{r}\int_0^{R} r' dr' \sqrt{y'}\Big|_{y'=(r-r')^2}^{y'=(r+r')^2}$$

(6) At this point we need to differentiate between the cases when $r < r'$ (observation point closer to the center of sphere than the particular source point r') or $r > r'$.

7 Let us first consider the case $\underline{\underline{r > R}}$ (observation point is outside the sphere). Thus r' is always less than r. In this case

$$\sqrt{(r-r')^2} = r - r' \qquad (\because r > r')$$



Using this in 5

$\underline{\underline{r > R}}$

$$\phi(r) = \frac{\rho_0}{2\epsilon_0}\,\frac{1}{r}\int_0^R r'\,dr'\left[(r+r') - (r-r')\right]$$

$$= \frac{\rho_0}{\epsilon_0}\,\frac{1}{r}\int_0^R r'^2\,dr'$$

$$= \frac{\rho_0}{\epsilon_0}\,\frac{1}{r}\,\frac{R^3}{3}$$

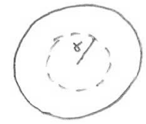
$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\,\rho_0\,\frac{4\pi R^3}{3}$$

using $Q = \rho_0 \frac{4\pi R^3}{3}$.

$$= \frac{1}{4\pi\epsilon_0}\,\frac{Q}{r}$$

8 Observe that for the case $r > R$ (observer outside the sphere) the observe measures the potential exactly that of a point charge Q placed at the center.

9 Next let us consider the case $r < R$, observation point inside the sphere. Now r' could be less than or greater than r.



$$\sqrt{(r-r')^2} = \begin{cases} r - r' & \text{if } r' < r \\ r' - r & \text{if } r < r' \end{cases}$$

To incorporate this we need to break the r' integral into two pieces.

$\underline{\underline{r < R}}$

$$\phi(\vec{r}) = \frac{\rho_0}{2\epsilon_0} \frac{1}{r} \int_0^R r' dr' \sqrt{y'} \Big|_{y'=(r-r')^2}^{y'=(r+r')^2}$$

$$= \frac{\rho_0}{2\epsilon_0} \frac{1}{r} \underbrace{\int_0^r r' dr' \sqrt{y'} \Big|_{y'=(r-r')^2}^{y'=(r+r')^2}}_{r' < r} + \frac{\rho_0}{2\epsilon_0} \frac{1}{r} \underbrace{\int_r^R r' dr' \sqrt{y'} \Big|_{y'=(r-r')^2}^{y'=(r+r')^2}}_{r < r'} .$$

$$= \frac{\rho_0}{2\epsilon_0} \frac{1}{r} \int_0^r r' dr' \left[(r+r') - (r-r') \right] + \frac{\rho_0}{2\epsilon_0} \frac{1}{r} \int_r^R r' dr' \left[(r+r') - (r'-r) \right]$$

$$= \frac{\rho_0}{\epsilon_0} \frac{1}{r} \int_0^r r'^2 dr' + \frac{\rho_0}{\epsilon_0} \int_r^R r' dr'$$

$$= \frac{\rho_0}{\epsilon_0} \frac{1}{r} \frac{r^3}{3} + \frac{\rho_0}{\epsilon_0} \left[\frac{R^2}{2} - \frac{r^2}{2} \right]$$

$$= \frac{\rho_0}{\epsilon_0} \left[\frac{R^2}{2} - \frac{r^2}{6} \right]$$

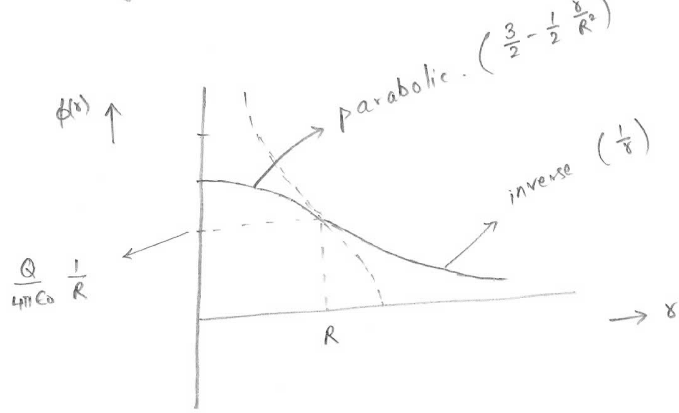
$$= \frac{1}{\epsilon_0} \frac{Q}{\frac{4\pi}{3} R^3} \left[\frac{R^2}{2} - \frac{r^2}{6} \right] \qquad \left(\text{using } \rho_0 \frac{4\pi}{3} R^3 = Q \right)$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{R} \left[\frac{3}{2} - \frac{1}{2} \frac{r^2}{R^2} \right] .$$

10 Thus, using 7 and 9 we have the electric potential of a uniformly charged solid sphere of radius R,

$$\phi(\vec{r}) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r} & R < r \\[2ex] \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{R}\left[\dfrac{3}{2} - \dfrac{1}{2}\dfrac{r^2}{R^2}\right] & r < R \end{cases}$$

11 Plotting $\phi(r)$ with respect to r.



12 Let us now calculate the electric field.

$$\vec{E}(\vec{r}) = - \vec{\nabla} \phi(\vec{r})$$

$$= - \vec{\nabla} \begin{cases} \frac{Q}{4\pi\epsilon_0} \frac{1}{r} & R < r \\ \frac{Q}{4\pi\epsilon_0} \frac{1}{R} \left[\frac{3}{2} - \frac{1}{2} \frac{r^2}{R^2} \right] & r < R \end{cases}$$

$$= \begin{cases} - \frac{Q}{4\pi\epsilon_0} \vec{\nabla} \frac{1}{r} & R < r \\ + \frac{Q}{4\pi\epsilon_0} \frac{1}{2R^3} \vec{\nabla} . r^2 & r < R \end{cases}$$

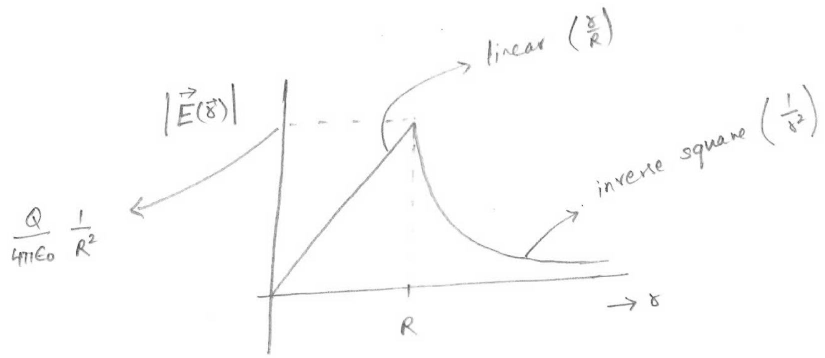
13 $$\vec{\nabla} \frac{1}{r} = - \frac{1}{r^2} \vec{\nabla} r = - \frac{\hat{r}}{r^2}$$

$$\vec{\nabla} r^2 = 2r \vec{\nabla} r = 2r\hat{r}$$

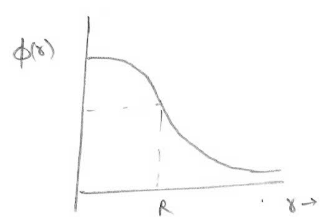
14 Using 13 in 12

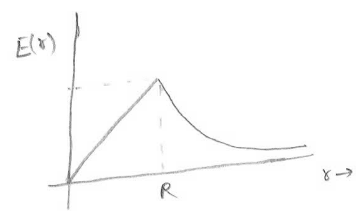
$$\vec{E}(\vec{r}) = \begin{cases} \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} & R < r \\ \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \frac{r}{R} \hat{r} & r < R \end{cases}$$

(15) Plot of magnitude of $\vec{E}(\vec{r})$ w.r.t. r.



(16) Compare the plots of electric field and electric potential.





They are related. $\left(\vec{E} = -\vec{\nabla}\phi\right)$

$$E(r) = -\frac{\partial}{\partial r}\phi(r)$$

(17) Comments

(i) Particle falling inside a tunnel through Earth.

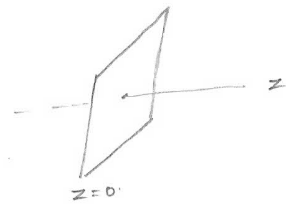
(ii) Rutherford's experiment.

(iii) Electric breakdown of vacuum and radius of electron.

II Surface charge

① Consider a uniformly charged plane of infinite extent with surface charge density σ.

$$\rho(\vec{r}) = \sigma \, \delta(z)$$



② The electric potential is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \, \frac{\sigma \, \delta(z')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \, \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}}$$

$x - x' = x''$ $\quad -dx' = dx''$ $\quad x' = \pm\infty \Rightarrow x'' = \mp\infty$

$y - y' = y''$ $\quad -dy' = dy''$ $\quad y' = \pm\infty \Rightarrow y'' = \mp\infty$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx'' \int_{-\infty}^{+\infty} dy'' \, \frac{1}{\sqrt{x''^2 + y''^2 + z^2}}$$

(3) Using cylindrical coordinates we have.

$$x'' = r'' \cos\phi'' \qquad dx''\,dy'' = r''\,dr''\,d\phi''$$
$$y'' = r'' \sin\phi''$$

(4) $$\phi(\vec{r}) = \frac{\sigma}{4\pi\epsilon_0}\int_0^\infty r''\,dr''\int_0^{2\pi} d\phi''\,\frac{1}{\sqrt{r''^2+z^2}}$$

$$= \frac{\sigma}{4\pi\epsilon_0}\,2\pi\int_0^\infty r''\,dr''\,\frac{1}{\sqrt{r''^2+z^2}}$$

$$r''^2 + z^2 = y'' \qquad r''=0 \Rightarrow y'' = z^2$$
$$2r''\,dr'' = dy'' \qquad r''=\infty \Rightarrow y''=\infty$$

$$\frac{y''^{(-\frac{1}{2}+1)}}{(-\frac{1}{2}+1)} = 2\sqrt{y''}$$

$$= \frac{\sigma}{2\epsilon_0}\int_{z^2}^\infty \frac{dy''}{2}\,\frac{1}{\sqrt{y''}}$$

R — radius of disc such that $z << R$.

$$= \frac{\sigma}{2\epsilon_0}\,\sqrt{y''}\,\Big|_{y''=z^2}^{y''=(R^2+z^2)\to\infty}$$

$$= \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+z^2} - |z|\right]$$

$$= \frac{\sigma}{2\epsilon_0}\,R\left[\sqrt{1+\frac{z^2}{R^2}} - \frac{|z|}{R}\right]$$

__Note__: If we consider a finite disc from the start then $x-x'=x''$ substitution can be done only if our observation point is on the z-axis.

(5) For $|z| << R$ we have.

$$\phi(\vec{r}) = \frac{\sigma}{2\epsilon_0}\left[R - |z|\right]$$

(keeping only the leading order in $\frac{z}{R}$.)