

Gauss's law

$$\textcircled{1} \quad \vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \rho(\vec{r})$$

$$\int_V d^3r \quad \vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \int_V d^3r \rho(\vec{r})$$

$$\int_S d\vec{a} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} Q_{en}$$

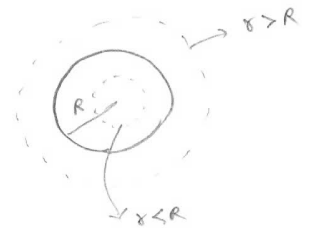
V - a closed volume.

S - surface enclosing volume V .

Q_{en} - total charge in volume V .

② Example ① : Uniformly charged solid sphere of radius R with charge Q .
Using symmetry argue that $\vec{E}(\vec{r})$ is normal to the surface.

$$E (4\pi r^2) = \frac{1}{\epsilon_0} Q_{en}$$



$r > R$:

$$E (4\pi r^2) = \frac{1}{\epsilon_0} Q$$

$$E = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}$$

$$\frac{Q_{en}}{Q} = \frac{\cancel{4\pi} \frac{4\pi}{3} r^3}{\cancel{4\pi} \frac{4\pi}{3} R^3}$$

$r < R$

$$E (4\pi r^2) = \frac{1}{\epsilon_0} Q \frac{r^3}{R^3}$$

$$E = \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3}$$

Thus we have:

$$\vec{E}(\vec{r}) = \begin{cases} \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} & r > R \\ \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \frac{r}{R} \hat{r} & r < R \end{cases}$$

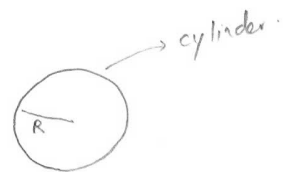
$r > R$

$r < R$

③ Example ②: Uniformly charged cylinder of radius R with charge per unit length λ .

Using Gauss's law we have.

$$E (2\pi r L) = \frac{1}{\epsilon_0} Q_{\text{en.}}$$



$r > R$:

$$E (2\pi r L) = \frac{1}{\epsilon_0} \lambda L$$

$$\lambda = \frac{Q}{L}$$

L - (Infinite) length of cylinder.

$$E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r}$$

$$\frac{Q_{\text{en.}}}{Q} = \frac{\lambda L \pi r^2}{\lambda L \pi R^2}$$

$r < R$:

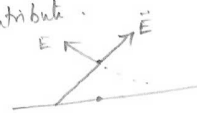
$$E (2\pi r L) = \frac{1}{\epsilon_0} Q \frac{r^2}{R^2}$$

$$E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{R} \frac{r}{R}$$

Thus, we have:

$$\vec{E}(r) = \begin{cases} \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r} \hat{r} & r > R \\ \frac{\lambda}{2\pi \epsilon_0} \frac{1}{R} \frac{r}{R} \hat{r} & r < R \end{cases}$$

→ Also, argue that the faces of cylinder do not contribute.

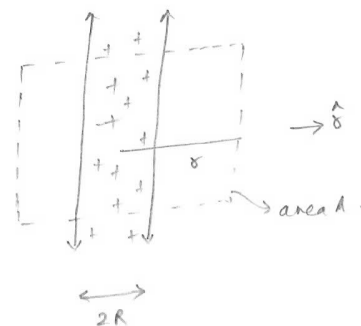


horizontal components cancel.

④ Example ③ : Uniformly charged solid slab of thickness R and surface charge density σ .

Using Gauss's law we have.

$$E(2A) = \frac{1}{\epsilon_0} Q_{en}$$



$\delta > R$

$$E(2A) = \frac{1}{\epsilon_0} Q$$

$$E = \frac{\sigma}{2\epsilon_0}$$

$$\frac{Q}{A} = \sigma$$

$\delta < R$

$$E(2A) = \frac{1}{\epsilon_0} Q_{en}$$

$$E(2A) = \frac{1}{\epsilon_0} Q \frac{\delta}{R}$$

$$E = \frac{\sigma}{2\epsilon_0} \frac{\delta}{R}$$

$$\frac{Q_{en}}{Q} = \frac{\sigma A \delta}{\sigma A 2R}$$

Thus, we have.

$$\vec{E}(\delta) = \begin{cases} \frac{\sigma}{2\epsilon_0} \hat{\delta} & \delta > R \\ \frac{\sigma}{2\epsilon_0} \frac{\delta}{R} \hat{\delta} & \delta < R \end{cases}$$

⑤ Sphere integral more explicitly is

$$\begin{aligned}
 \oint_{\text{sphere}} d\vec{a} \cdot \vec{E}(\vec{r}) &= \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta R^2 \hat{r} \cdot \vec{E}(\vec{r}) \\
 &= R^2 \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta E_r(R, \theta, \phi) \\
 &= R^2 E_r(R) \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \quad \rightarrow \text{symmetry argument} \\
 &= R^2 E_r(R) 4\pi \\
 &= E_r(R) [4\pi R^2]
 \end{aligned}$$

Area element
 $d\vec{a} = (R d\theta \hat{\theta}) \times (R \sin\theta d\phi \hat{\phi})$
 $= \hat{r} R^2 \sin\theta d\theta d\phi$

⑥ Cylinder integral more explicitly is

$$\begin{aligned}
 \oint_{\text{cylinder}} d\vec{a} \cdot \vec{E}(\vec{r}) &= \int_{\text{long surface}} d\vec{a} \cdot \vec{E}(\vec{r}) + \int_{\text{circular faces}} d\vec{a} \cdot \vec{E}(\vec{r}) \\
 &= \iint (R d\phi \hat{\phi} \times d\hat{z}) \cdot \vec{E}(\vec{r}) + \iint (R d\phi \hat{\phi} \times d\hat{z}) \cdot \vec{E}(\vec{r}) \\
 &= \iint R d\phi dz \hat{r} \cdot \vec{E}(\vec{r}) + \iint R d\phi dz \underbrace{(-\hat{z}) \cdot \vec{E}(\vec{r})}_{=0} \\
 &= R \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dz E_r(R, \phi, z) \\
 &= R E_r(R) \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dz \quad \rightarrow \text{symmetry argument} \\
 &= E_r(R) [2\pi R L]
 \end{aligned}$$

⑦ Example ④ : Solid cylinder of radius R with charge density

$$\rho(\vec{r}) = \begin{cases} k r & r < R \\ 0 & r > R \end{cases}$$

Total charge is

$$\begin{aligned} Q &= \int d^3r \rho(\vec{r}) \\ &= \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dz \int_0^R r dr k r \\ &= 2\pi L k \frac{R^3}{3} \end{aligned}$$

$$\int_{-\infty}^{+\infty} dz = L$$

$$\frac{Q}{L} = \lambda$$

$$k = \frac{Q}{\left(\frac{2\pi L}{3} R^3\right)}$$

$r > R$:

$$E (2\pi r L) = \frac{1}{\epsilon_0} Q$$

$$E = \frac{Q}{2\pi \epsilon_0 L} \frac{1}{r} = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r}$$

$r < R$:

$$E (2\pi r L) = \frac{1}{\epsilon_0} Q_{en.}$$

$$E (2\pi r L) = \frac{1}{\epsilon_0} Q \frac{r^3}{R^3}$$

$$E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{R} \frac{r^2}{R^2}$$

$$\frac{Q_{en.}}{Q} = \frac{\left(\frac{2\pi L k}{3} \frac{r^3}{3}\right)}{\left(\frac{2\pi L k}{3} \frac{R^3}{3}\right)}$$

Thus,

$$\vec{E}(\vec{r}) = \begin{cases} \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r} \hat{r} & r > R \\ \frac{\lambda}{2\pi \epsilon_0} \frac{1}{R} \frac{r^2}{R^2} \hat{r} & r < R \end{cases}$$

⑧ Example ⑤: Solid cylinder of radius R with

charge density

$$\rho(r) = \begin{cases} k r^n & r < R \\ 0 & r > R \end{cases}$$

Total charge

$$Q = \int d^3r \rho(r) = \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dz \int_0^R r dr k r^n = 2\pi L k \frac{R^{n+2}}{(n+2)}$$

$$\int_{-\infty}^{+\infty} dz = L$$

$$\frac{Q}{L} = \lambda$$

$$k = \frac{\lambda}{\left[\frac{2\pi}{n+2} R^{n+2} \right]}$$

$r > R$:

$$E(2\pi r L) = \frac{1}{\epsilon_0} Q$$

$$E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r}$$

$$\frac{Q_{en}}{Q} = \frac{2\pi L k \left(\frac{r^{n+2}}{n+2} \right)}{2\pi L k \left(\frac{R^{n+2}}{n+2} \right)}$$

$r < R$:

$$E(2\pi r L) = \frac{1}{\epsilon_0} Q_{en}$$

$$E(2\pi r L) = \frac{1}{\epsilon_0} \frac{r^{n+2}}{R^{n+2}} Q$$

$$E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{R} \left(\frac{r}{R} \right)^{n+1}$$

Thus,

$$\vec{E}(\vec{r}) = \begin{cases} \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r} & r > R \\ \frac{\lambda}{2\pi \epsilon_0} \frac{1}{R} \left(\frac{r}{R} \right)^{n+1} & r < R \end{cases}$$

⑨ Example ⑥: Cylinder with charge density

$$\rho(\vec{r}) = \frac{k}{r} \quad a \leq r \leq b \quad \text{and } 0 \text{ otherwise.}$$

Total charge:

$$\begin{aligned} Q &= \int d^3r \rho(\vec{r}) \\ &= 2\pi L \int_a^b r dr \frac{k}{r} \\ &= 2\pi L k (b-a) \end{aligned}$$

$$\int_{-\infty}^{\infty} dz = L$$

$$\frac{Q}{L} = \lambda$$

$$k = \frac{Q}{2\pi L(b-a)}$$

$r > b$:

$$E(2\pi r L) = \frac{1}{\epsilon_0} Q \Rightarrow E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r}$$

$$\Rightarrow E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r}$$

$r < a$:

$$E(2\pi r L) = \frac{1}{\epsilon_0} Q_{\text{en}} = 0 \Rightarrow E = 0$$

$$\Rightarrow E = 0$$

$a < r < b$

$$E(2\pi r L) = \frac{1}{\epsilon_0} Q_{\text{en}}$$

$$E(2\pi r L) = \frac{1}{\epsilon_0} Q \left(\frac{r-a}{b-a} \right)$$

$$E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r} \left(\frac{r-a}{b-a} \right)$$

$$\frac{Q_{\text{en}}}{Q} = \frac{2\pi L k (r-a)}{2\pi L k (b-a)}$$

Thus, we have.

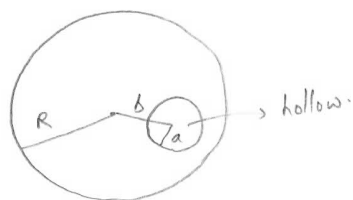
$$\vec{E}(\vec{r}) = \begin{cases} 0 & r < a \\ \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r} \left(\frac{r-a}{b-a} \right) & a < r < b \\ \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r} & b < r \end{cases}$$

$r < a$

$a < r < b$

$b < r$

- ⑩ Example ⑦: Solid sphere of radius R (uniform charge) with a smaller sphere removed out.



- E due to sphere R is known → E_1
- E due to sphere a is known → E_2
- Use superposition principle. → E_{answer}

$$E_{\text{answer}} = E_1 - E_2$$