SET-12

Conservation of energy

① Let us remove the contraint of electrostatics and consider the complete Maxwell's equations

$$\vec{D} = \epsilon_0 \vec{E}$$
$$\vec{B} = \mu_0 \vec{H}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_e$$
$$-\vec{\nabla} \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0 \rightarrow \vec{j}_m$$
$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \rho_m$$
$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}_e$$

$\rho_m \rightarrow$ magnetic monople charge density

$\vec{j}_m \rightarrow$ magnetic monople current density

② The Lorentz force is given by

$$\vec{F} = q_e \left[\vec{E} + \vec{v} \times \vec{B} \right] + q_m \left[\vec{H} - \vec{v} \times \vec{D} \right]$$

③ where the negative sign is suggested by the invariance under

$$\rho_e \rightarrow \rho_m \qquad \rho_m \rightarrow -\rho_e$$
$$\vec{j}_e \rightarrow \vec{j}_m \quad \text{and} \quad \vec{j}_m \rightarrow -\vec{j}_e$$
$$\vec{E} \rightarrow \vec{H} \qquad \vec{H} \rightarrow -\vec{E}$$

④ Note that

$$\text{Power} = \frac{\text{Energy}}{\text{time}} = \frac{Fd}{t} \rightarrow \vec{F} \cdot \vec{v}$$

(5) Thus, the rate of energy transfered from the electromagnetic field to the charge is given by

$$\vec{F}\cdot\vec{V} = q_e\left[\vec{E} + \vec{V}\times\vec{B}\right]\cdot\vec{V} + q_m\left[\vec{H} - \vec{V}\times\vec{D}\right]\cdot\vec{V}$$

$$= (q_e\vec{V})\cdot\vec{E} + (q_m\vec{V})\cdot\vec{H}$$

$$= \int d^3r\left[\vec{j}_e\cdot\vec{E} + \vec{j}_m\cdot\vec{H}\right]$$

where used the charge density of a particle as

$$\vec{j}_e(\vec{r}) = q_e\,\vec{V}_a\,\delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$\vec{j}_m(\vec{r}) = q_m\,\vec{V}_a\,\delta^{(3)}(\vec{r} - \vec{r}_a(t)).$$

(6) Using Maxwell's equation we have.

$$\frac{\partial}{\partial t}\left(\frac{1}{2}x^2\right) = x\frac{\partial x}{\partial t}$$

$$\vec{j}_e\cdot\vec{E} = \left[\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t}\right]\cdot\vec{E}$$

$$= (\vec{\nabla}\times\vec{H})\cdot\vec{E} - \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2\right)$$

$$\vec{j}_m\cdot\vec{H} = \left[-\vec{\nabla}\times\vec{E} - \frac{\partial\vec{B}}{\partial t}\right]\cdot\vec{H}$$

$$= -(\vec{\nabla}\times\vec{E})\cdot\vec{H} - \frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0 H^2\right)$$

(7) Adding we have

$$\vec{j}_e\cdot\vec{E} + \vec{j}_m\cdot\vec{H} + \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) = (\vec{\nabla}\times\vec{H})\cdot\vec{E} - (\vec{\nabla}\times\vec{E})\cdot\vec{H}.$$

8 $$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \epsilon_{ijk}\,\nabla_i(E_j H_k)$$
$$= \epsilon_{ijk}(\nabla_i E_j)H_k + \epsilon_{ijk}E_j(\nabla_i H_k)$$
$$= (\vec{\nabla}\times\vec{E})\cdot\vec{H} - (\vec{\nabla}\times\vec{H})\cdot\vec{E}.$$

9 Using 8 in 7 we have

$$\frac{\partial}{\partial t}U + \vec{\nabla}\cdot\vec{S} + \vec{j}_e\cdot\vec{E} + \vec{j}_m\cdot\vec{H} = 0$$

where

$U = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2$ $\longrightarrow$ electromagnetic field energy density

$\vec{S} = \vec{E}\times\vec{H}$ $\longrightarrow$ flux of electromagnetic energy (Poynting vector)

10 Integrating over a volume V we have.

$$\frac{\partial}{\partial t}\left(\int_V d^3r\, U\right) + \oint_S d\vec{a}\cdot\vec{S} + \int_V d^3r\, \vec{j}_e\cdot\vec{E} + \int_V d^3r\, \vec{j}_m\cdot\vec{H} = 0$$

rate of change of electromagnetic field energy inside volume V.

rate at which electromagnetic field energy is leaving volume V.

rate of transfer of electromagnetic field energy to the charges.

11 What is the interpretation in electrostatics?

$$\vec{S} = 0$$

$$U = \frac{1}{2}\epsilon_0 E^2$$

$$W = \rho_e \phi$$

such that we have.

$$\frac{1}{2}\epsilon_0 E^2 + \rho_e \phi = \text{constant}$$

and in integrated form we have.

$$\underbrace{\int_V d^3r \left(\frac{1}{2}\epsilon_0 E^2\right)}_{\text{electromagnetic field energy inside volume } V} + \underbrace{q_a \phi(\vec{r}_a)}_{\text{work done on the charges to bring them from infinity.}} = \text{const}$$

Energy in electrostatics

(1) $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$

$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\phi$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$\phi \rightarrow$ Electric potential

$\vec{E} \rightarrow$ Electric field

(2) Bring a test charge "q"

$$\vec{F} = q_a \vec{E}(\vec{r}_a)$$

$$U = q_a \phi(\vec{r}_a)$$

$\vec{F} \rightarrow$ force on charge q

$U \rightarrow$ interaction energy between charge q and charge distribution $\rho(\vec{r})$.

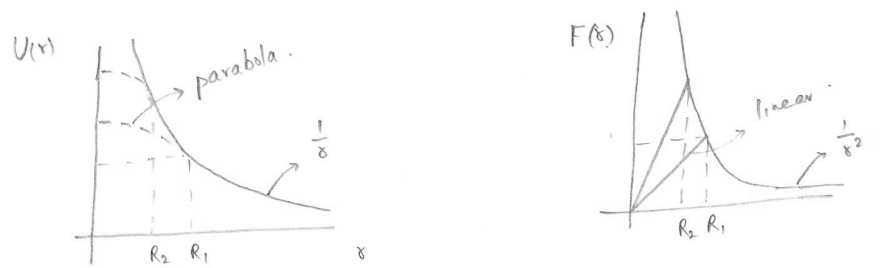
(3) Relation between force and energy

$$\begin{aligned} \vec{F} &= q_a \vec{E}(\vec{r}_a) \\ &= -q_a \vec{\nabla} \phi(\vec{r}_a) \\ &= -\vec{\nabla} \left[q_a \phi(\vec{r}_a) \right] \\ &= -\vec{\nabla} U \end{aligned}$$

4 Let us analyse the Rutherford scattering experiment.

$$U(r) = q\,\phi(\vec{r}) = \begin{cases} \dfrac{qQ}{4\pi\epsilon_0}\dfrac{1}{r} & r > R \\[2ex] \dfrac{qQ}{4\pi\epsilon_0}\dfrac{1}{R}\left[\dfrac{3}{2} - \dfrac{1}{2}\dfrac{r^2}{R^2}\right] & r < R \end{cases}$$

$$\vec{F}(\vec{r}) = q\,\vec{E}(\vec{r}) = \begin{cases} \dfrac{qQ}{4\pi\epsilon_0}\dfrac{1}{r^2}\hat{r} & r > R \\[2ex] \dfrac{qQ}{4\pi\epsilon_0}\dfrac{1}{R^2}\dfrac{r}{R}\hat{r} & r < R. \end{cases}$$



Kinetic energy of a projectile can probe the radius of a charged sphere.

— if projectile rips through: $\dfrac{qQ}{4\pi\epsilon_0}\dfrac{1}{R}\dfrac{3}{2} < \dfrac{1}{2}mv^2$

$$\Rightarrow \left(\frac{qQ}{4\pi\epsilon_0}\frac{3}{2}\right)\left(\frac{1}{2}mv^2\right)^{-1} < R$$

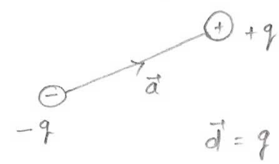
— if projectile bounces back: $\dfrac{qQ}{4\pi\epsilon_0}\dfrac{1}{R}\dfrac{3}{2} > \dfrac{1}{2}mv^2$

$$\Rightarrow \left(\frac{qQ}{4\pi\epsilon_0}\frac{3}{2}\right)\left(\frac{1}{2}mv^2\right)^{-1} > R$$

5 Point particle

$$\vec{F} = q_a \vec{E}(\vec{r}_a)$$
$$= -\vec{\nabla} U$$
$$U = q_a \phi(\vec{r}_a)$$

6 Electric dipole moment :



$$\vec{d} = q\vec{a}$$

$$\vec{r}_+ = \vec{r}_0 + \frac{\vec{a}}{2}$$
$$\vec{r}_- = \vec{r}_0 - \frac{\vec{a}}{2}$$

$$\vec{F} = q\vec{E}(\vec{r}_+) - q\vec{E}(\vec{r}_-)$$
$$= q\vec{E}\left(\vec{r}_0 + \frac{\vec{a}}{2}\right) - q\vec{E}\left(\vec{r}_0 - \frac{\vec{a}}{2}\right)$$
$$= \cancel{q\vec{E}(\vec{r}_0)} + q\frac{\vec{a}}{2}\cdot\vec{\nabla}\vec{E}(\vec{r}_0) + \cdots$$
$$\quad \cancel{-q\vec{E}(\vec{r}_0)} + q\frac{\vec{a}}{2}\cdot\vec{\nabla}\vec{E}(\vec{r}_0) + \cdots$$

$\|\vec{a}\cdot\vec{\nabla}\vec{E}\| \ll \|\vec{E}\|$

$a \ll \lambda$

$\lambda \to$ characteristic wavelength.

$$= \vec{d}\cdot\vec{\nabla}\vec{E}(\vec{r}_0)$$
$$= -\vec{d}\cdot\vec{\nabla}\vec{\nabla}\phi \qquad \text{(using } \vec{E} = -\vec{\nabla}\phi \text{ — electrostatics)}$$
$$= -\vec{\nabla}\left(\vec{d}\cdot\vec{\nabla}\phi\right)$$
$$= \vec{\nabla}\,\vec{d}\cdot\vec{E}$$
$$= -\vec{\nabla}\,U_{dipole}$$

$$\boxed{U_E = -\vec{d}\cdot\vec{E}}$$

7 Torque on electric dipole

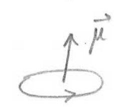
$$\vec{\tau}_E = \sum_a (\vec{r}_a - \vec{r}_0) \times \vec{F}(\vec{r}_a)$$

$$= q\,\frac{\vec{a}}{2} \times \vec{E}(\vec{r}_+) - q\left(-\frac{\vec{a}}{2}\right) \times \vec{E}(\vec{r}_-)$$

$$= q\,\frac{\vec{a}}{2} \times \vec{E}(\vec{r}_0) + q\,\frac{\vec{a}}{2} \times \vec{E}(\vec{r}_0)$$

$$= \vec{d} \times \vec{E}$$

8 Magnetic dipole moment



$$\vec{\mu} = \frac{1}{2}\, q\, \vec{r}_a \times \vec{V}_a = \frac{q}{2m}\vec{L}$$

$$\vec{\mu} = I A\, \hat{n}$$

$$\mu = \frac{1}{2} q\, r\, v$$

$$= \frac{1}{2}\frac{q}{\Delta t}\, r\, \underbrace{\frac{\Delta x}{2\pi r}}_{N}\, 2\pi r$$

$$= I\, \pi r^2$$

$$= I A$$

$$U_B = -\vec{\mu} \cdot \vec{B}$$

$$\vec{\tau}_B = \vec{\mu} \times \vec{B}$$