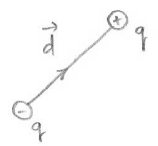
SET-13

Macroscopic electrodynamics

(1) Electric dipole: The simplest example is that of two equal and opposite charges separated by distance a.



$$U_E = -\vec{d}\cdot\vec{E} \quad \rightarrow \text{energy}$$

$$\vec{F}_E = -\vec{\nabla}\, U_E = \vec{\nabla}(\vec{d}\cdot\vec{E}) \quad \rightarrow \text{force}$$

$$\vec{\tau}_E = \vec{d}\times\vec{E} \quad \rightarrow \text{torque}$$

(2) Magnetic dipole: The simplest example is a bar magnet (North pole and South pole). Another example is a circular current loop carrying current.

$$U_B = -\vec{\mu}\cdot\vec{B} \quad \rightarrow \text{energy}$$

$$\vec{F}_B = -\vec{\nabla}\, U_B = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) \quad \rightarrow \text{force}$$

$$\vec{\tau}_B = \vec{\mu}\times\vec{B} \quad \rightarrow \text{torque}$$

(3) Let us consider a neutral atom with charges e_a positioned at $\vec{r}_a$. Let $\vec{R}$ be the center-of-charge and $\vec{V}$ the velocity of the center-of-charge.

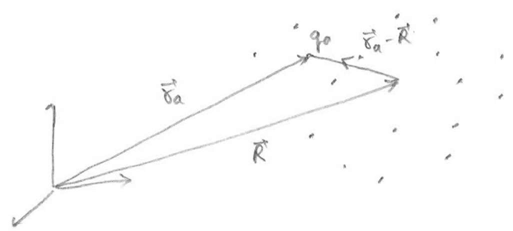
Let us define.

$$\sum_a e_a = 0$$

$$\vec{d} = \sum_a e_a (\vec{r}_a - \vec{R}) = \sum_a e_a \vec{r}_a$$

$$\vec{\mu} = \sum_a e_a (\vec{r}_a - \vec{R}) \times (\vec{v}_a - \vec{V})$$

(4)



The total force on the atom in the presence of an electric and magnetic field is

$$\vec{F} = \sum_a \left[e_a \vec{E}(\vec{r}_a) + e_a \vec{v}_a \times \vec{B}(\vec{r}_a) \right]$$

5 Taylor expansion of the fields around the center of charge yields.

$$\vec{E}(\vec{r}_a) = \vec{E}(\vec{R}) + \left[(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}\right] \vec{E}(\vec{R}) + \cdots$$

$$\vec{B}(\vec{r}_a) = \vec{B}(\vec{R}) + \left[(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}\right] \vec{B}(\vec{R}) + \cdots$$

6 Approximations:

(i) $\left|(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}\right| \to \left|(\vec{r}_a - \vec{R}) \cdot i\vec{k}_0\right| << 1$

where $\lambda_0 = \frac{2\pi}{|\vec{k}_0|}$ is a characteristic wavelength of the atom.

(ii) $|\vec{V}| << |\vec{V}_a| << c$.

7 Using 5 in 4

$$\vec{F} = \underbrace{\sum_a e_a}_{=0} \vec{E}(\vec{R}) + \overbrace{\sum_a e_a (\vec{r}_a - \vec{R})}^{\vec{d}} \cdot \vec{\nabla} \vec{E}(\vec{R})$$

$$+ \underbrace{\sum_a e_a \vec{V}_a}_{\frac{d}{dt}\vec{d}} \times \vec{B}(\vec{R}) + \sum_a e_a \vec{V}_a \times \left[(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}\right] \vec{B}(\vec{R})$$

8 Thus, we have

$$\vec{F} = \vec{d}\cdot\vec{\nabla}\,\vec{E} + \left(\frac{d}{dt}\vec{d}\right)\times\vec{B} + \sum_a e_a \vec{v}_a \times [(\vec{r}_a - \vec{R})\cdot\vec{\nabla}]\,\vec{B}$$

9 without giving the details, which will be presented elsewhere, we calculate the third term within the approximation in 6 as

$$\sum_a e_a \vec{v}_a \times [(\vec{r}_a - \vec{R})\cdot\vec{\nabla}]\,\vec{B} \longrightarrow \vec{\mu}\times(\vec{\nabla}\times\vec{B}) + (\vec{\mu}\cdot\vec{\nabla})\,\vec{B} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$$

10 Using 9 in 8

$$\vec{F} = \vec{d}\cdot\vec{\nabla}\,\vec{E} + \frac{d}{dt}(\vec{d}\times\vec{B}) - \vec{d}\times\left(\frac{d\vec{B}}{dt}\right) + \vec{\mu}\times(\vec{\nabla}\times\vec{B}) + (\vec{\mu}\cdot\vec{\nabla})\,\vec{B}$$

$$= \vec{d}\cdot\vec{\nabla}\,\vec{E} + \frac{d}{dt}(\vec{d}\times\vec{B}) + \vec{d}\times(\vec{\nabla}\times\vec{E}) + \vec{\mu}\times(\vec{\nabla}\times\vec{B}) + (\vec{\mu}\cdot\vec{\nabla})\,\vec{B}.$$

where we used $\frac{d}{dt} \sim \frac{\partial}{\partial t}$, which is true for $|\vec{V}| \ll |\vec{V}_a|$.

(11) Thus, we have the force on an atom to be

$$\vec{F}_{atom} = \frac{d}{dt}(\vec{d}\times\vec{B}) + \vec{d}\times(\vec{\nabla}\times\vec{E}) + (\vec{d}\cdot\vec{\nabla})\vec{E} + \vec{\mu}\times(\vec{\nabla}\times\vec{B}) + (\vec{\mu}\cdot\vec{\nabla})\vec{B}$$

(12) The total force on a macroscopic body will be

$$\vec{F} = \int d^3r \; n(\vec{r}) \; \vec{F}_{atom}(\vec{r}),$$

where $n(\vec{r})$ is the number density per unit volume.

(13) Using (11) in (12)

$$\vec{F} = \int d^3r \; n(\vec{r}) \left[\frac{d}{dt}(\vec{d}\times\vec{B}) + \vec{d}\times(\vec{\nabla}\times\vec{E}) + (\vec{d}\cdot\vec{\nabla})\vec{E} + \vec{\mu}\times(\vec{\nabla}\times\vec{B}) + (\vec{\mu}\cdot\vec{\nabla})\vec{B} \right]$$

(14) Define

$$\vec{P}(\vec{r},t) = n(\vec{r})\;\vec{d}(\vec{r},t)$$

$$\vec{M}(\vec{r},t) = n(\vec{r})\;\vec{\mu}(\vec{r},t)$$

$\vec{P}$ – electric polarization

$\vec{M}$ – magnetic polarization

15 Using 14 in 13

$$\vec{F} = \int d^3r \left[\frac{d}{dt}(\vec{P}\times\vec{B}) + \vec{P}\times(\vec{\nabla}\times\vec{E}) + (\vec{P}\cdot\vec{\nabla})\vec{E} + \vec{M}\times(\vec{\nabla}\times\vec{B}) + (\vec{M}\cdot\vec{\nabla})\vec{B} \right]$$

16 Using $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ and $\frac{\partial}{\partial t} \approx \frac{d}{dt}$ (true for $|\vec{v}| << |\vec{v}_0| << c$)

we have.

$$\frac{d}{dt}(\vec{P}\times\vec{B}) + \vec{P}\times(\vec{\nabla}\times\vec{E}) = \frac{\partial}{\partial t}(\vec{P}\times\vec{B}) - \vec{P}\times\left(\frac{\partial\vec{B}}{\partial t}\right)$$

$$= \left(\frac{\partial\vec{P}}{\partial t}\right)\times\vec{B}.$$

17 Further,

$$\vec{M}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{M})$$

$$= \vec{M}\,\vec{\nabla}\,\vec{B} - (\vec{M}\cdot\vec{\nabla})\vec{B} + \vec{B}\,\vec{\nabla}\,\vec{M} - (\vec{B}\cdot\vec{\nabla})\vec{M}$$

$$= \vec{\nabla}(\vec{M}\cdot\vec{B}) - (\vec{M}\cdot\vec{\nabla})\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{M}$$

18 Using 16 and 17 in 15

$$\vec{F} = \int d^3r \left[(\vec{P}\cdot\vec{\nabla})\vec{E} + \left(\frac{\partial\vec{P}}{\partial t}\right)\times\vec{B} + (\vec{\nabla}\times\vec{M})\times\vec{B} + \vec{\nabla}(\vec{M}\cdot\vec{B}) - (\vec{B}\cdot\vec{\nabla})\vec{M} \right]$$

(19) On the boundary of the macroscopic body we have $n(\vec{r}) = 0$, thus $\vec{P}$ and $\vec{M}$ are zero on the boundary. Thus, we have.

$$\int d^3r \ \vec{\nabla}(\vec{M}\cdot\vec{B}) = 0$$

$$\int d^3r \ (\vec{P}\cdot\vec{\nabla})\,\vec{E} = \underbrace{\int d^3r \ \vec{\nabla}\cdot(\vec{P}\,\vec{E})}_{=0} - \int d^3r \ (\vec{\nabla}\cdot\vec{P})\,\vec{E}$$

$$= -\int d^3r \ (\vec{\nabla}\cdot\vec{P})\,\vec{E}$$

$$\int d^3r \ (\vec{B}\cdot\vec{\nabla})\,\vec{M} = \underbrace{\int d^3r \ \vec{\nabla}\cdot(\vec{B}\,\vec{M})}_{=0} - \underbrace{\int d^3r \ (\vec{\nabla}\cdot\vec{B})\,\vec{M}}_{=0}$$

$$= 0$$

(20) Using (19) in (18)

$$\vec{F} = \int d^3r \left[-(\vec{\nabla}\cdot\vec{P})\,\vec{E} + \left\{ \frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\times\vec{M} \right\} \times \vec{B} \right]$$

(21) Comparing (20) with the expression for the Lorentz force on a continuous distribution of charges,

$$\vec{F} = \int d^3r \left[\rho\,\vec{E} + \vec{j}\times\vec{B} \right]$$

we identify the effective charge densities and current densities

$$\rho_{eff}(\vec{r},t) = -\vec{\nabla}\cdot\vec{P}(\vec{r},t)$$

$$\vec{j}_{eff}(\vec{r},t) = \frac{\partial}{\partial t}\vec{P}(\vec{r},t) + \vec{\nabla}\times\vec{M}(\vec{r},t).$$

(22) Thus, the macroscopic Maxwell's equations will be

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E}) = \rho - \vec{\nabla} \cdot \vec{P}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B}\right) = \frac{\partial}{\partial t} (\epsilon_0 \vec{E}) + \vec{j} + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \vec{M}$$

(23) Define

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} + \vec{M}$$

(24) which leads to

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$