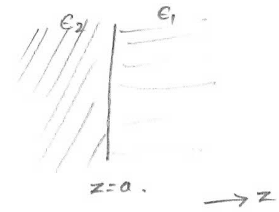
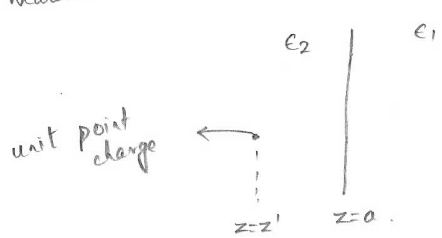
Green's function for a semi-infinite slab

1. We will here evaluate the Green's function for a semi-infinite dielectric slab described by

$$\epsilon(z) = \begin{cases} \epsilon_2 & z < a \\ \epsilon_1 & a < z \end{cases}$$



2. Let us consider the case. $z' < a$, which corresponds to finding the electric potential due to a unit point charge inside the dielectric material ϵ_2.



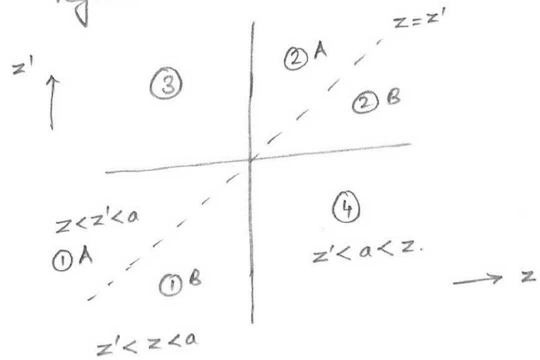
3. The reduced Green's function satisfies the equation

$$\left[-\frac{\partial}{\partial z}\epsilon(z)\frac{\partial}{\partial z} + \epsilon(z)k_\perp^2\right] g_e(z, z'; k_\perp) = \delta(z - z')$$

4 For the case $z' < a$, we have three distinct regions with solutions:

$$g_e(z,z';k_1) = \begin{cases} A\, e^{k_1 z} + B\, e^{-k_1 z} & z < z' < a \rightarrow 1A \\ C\, e^{k_1 z} + D\, e^{-k_1 z} & z' < z < a \rightarrow 1B \\ E\, e^{k_1 z} + F\, e^{-k_1 z} & z' < a < z \rightarrow 4 \end{cases}$$

5 The regions are conveniently visualized using



6 Continuity conditions at $z = z'$:

(i) $$g \Big|_{z=z'-\delta}^{z=z'+\delta} = 0$$

(ii) $$\epsilon \frac{\partial g}{\partial z} \Big|_{z=z'-\delta}^{z=z'+\delta} = -1$$

7 Continuity conditions at $z = a$:

(i) $$g \Big|_{z=a-\delta}^{z=a+\delta} = 0$$

(ii) $$\epsilon \frac{\partial g}{\partial z} \Big|_{z=a-\delta}^{z=a+\delta} = 0 .$$

8 We also have the boundary conditions :

(i) $g_\epsilon(z=+\infty, z'; k_\perp) = 0$

(ii) $g_\epsilon(z=-\infty, z'; k_\perp) = 0$

9 Six conditions in 6, 7 and 8 determine the six unknowns in the solution in 4 unambiguously.

10 Using 8-(i) and 8-(ii) we immediately have.

$$B = 0,$$

$$E = 0.$$

11 Using 7 we have

$$F e^{-k_\perp a} = C e^{k_\perp a} + D e^{-k_\perp a}.$$

$$-\cancel{k_\perp} \epsilon_1 F e^{-k_\perp a} = \epsilon_2 \cancel{k_\perp} C e^{k_\perp a} - \epsilon_2 \cancel{k_\perp} D e^{-k_\perp a}.$$

or

$$F e^{-k_\perp a} - C e^{k_\perp a} = D e^{-k_\perp a}.$$

$$F e^{-k_\perp a} + C \frac{\epsilon_2}{\epsilon_1} e^{k_\perp a} = D \frac{\epsilon_2}{\epsilon_1} e^{-k_\perp a}$$

$$F = \frac{\frac{\epsilon_2}{\epsilon_1} D + \frac{\epsilon_2}{\epsilon_1} D}{\frac{\epsilon_2}{\epsilon_1} + 1} = \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1} D.$$

$$C = \frac{\frac{\epsilon_2}{\epsilon_1} D e^{-2k_\perp a} - D e^{-2k_\perp a}}{\frac{\epsilon_2}{\epsilon_1} + 1} = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} D e^{-2k_\perp a}.$$

12 Using 10 and 11 in 4 we have

$$g_\epsilon(z,z';k_\perp) = \begin{cases} A\, e^{k_\perp z} & z<z'<a \quad \rightarrow 1A \\ \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} D\, e^{-2k_\perp a} e^{k_\perp z} + D e^{-k_\perp z} & z'<z<a \quad \rightarrow 1B \\ \frac{2\epsilon_2}{\epsilon_2+\epsilon_1} D\, e^{-k_\perp z} & z'<a<z \quad \rightarrow 4 \end{cases}$$

13 Using 6 we have.

$$\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} D\, e^{-2k_\perp a} e^{k_\perp z'} + D e^{-k_\perp z'} - A\, e^{k_\perp z'} = 0$$

$$\epsilon_2 k_\perp \left\{ \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} D\, e^{-2k_\perp a} e^{k_\perp z'} - D e^{-k_\perp z'} \right\} - \epsilon_2 k_\perp \left\{ A\, e^{k_\perp z'} \right\} = -1$$

which when subtracted together leads to

$$2D\, e^{-k_\perp z'} = \frac{1}{\epsilon_2 k_\perp}$$

$$D = \frac{1}{\epsilon_2} \frac{1}{2k_\perp} e^{+k_\perp z'}$$

14 Thus,

$$A = \left\{ \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{-2k_\perp a} + e^{-2k_\perp z'} \right\} \frac{1}{\epsilon_2} \frac{1}{2k_\perp} e^{k_\perp z'}$$

$$= \frac{1}{\epsilon_2} \frac{1}{2k_\perp} e^{-k_\perp z'} + \frac{1}{\epsilon_2} \frac{1}{2k_\perp} \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{k_\perp z' - 2k_\perp a}$$

15 Using 13 and 14 in 12

$$g_\epsilon(z,z';k_\perp) = \begin{cases} \frac{1}{\epsilon_2}\frac{1}{2k_\perp} e^{-k_\perp(z'-z)} + \frac{1}{\epsilon_2}\frac{1}{2k_\perp}\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{-k_\perp(a-z)} e^{-k_\perp(a-z')} & z<z'<a \\ \frac{1}{\epsilon_2}\frac{1}{2k_\perp} e^{-k_\perp(z-z')} + \frac{1}{\epsilon_2}\frac{1}{2k_\perp}\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{-k_\perp(a-z)} e^{-k_\perp(a-z')} & z'<z<a \\ \frac{1}{\epsilon_2}\frac{1}{2k_\perp}\frac{2\epsilon_2}{\epsilon_2+\epsilon_1} e^{-k_\perp(z-z')} & z'<a<z \end{cases}$$

$$= \frac{1}{\epsilon_2}\frac{1}{2k_\perp} e^{-k_\perp|z-z'|} + \frac{1}{\epsilon_2}\frac{1}{2k_\perp}\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{-k_\perp|z-a|} e^{-k_\perp|z'-a|}, \quad z'<a$$

16 Similarly, in homework, you show

$$g_\epsilon(z,z';k_\perp) = \frac{1}{\epsilon_1}\frac{1}{2k_\perp} e^{-k_\perp|z-z'|} + \frac{1}{\epsilon_1}\frac{1}{2k_\perp}\frac{\epsilon_1-\epsilon_2}{\epsilon_2+\epsilon_1} e^{-k_\perp|z-a|} e^{-k_\perp|z'-a|}, \quad a<z'.$$