Homework No. 02 (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Due date: Monday, 2013 Sep 9, 4.30pm

1. The Lorentz force law in SI units is
$$\mathbf{F} = q\big[\mathbf{E} + \mathbf{v} \times \mathbf{B}\big]. \tag{1}$$
Write down the Lorentz force law in Lorentz-Heaviside units.

2. In Gaussian units the power radiated by an accelerated charged particle of charge e is given by the Larmor formula,
$$P = \frac{2}{3}\frac{e^2}{c^3}\, a^2, \tag{2}$$
where a is the acceleration of the charged particle. Write down the Larmor formula in SI units, and in Lorentz-Heaviside units.

3. In Gaussian units the cyclotron frequency is
$$\omega_0 = \frac{eB}{mc}, \tag{3}$$
where m is the mass of electron. Write down the expression for cyclotron frequency in SI units, and in Lorentz-Heaviside units.

4. (Ref. Schwinger et al., problem 1, chapter 1.) For an arbitrarily moving charge, the charge and current densities are
$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}_a(t)) \tag{4}$$
and
$$\mathbf{j}(\mathbf{r}, t) = q\mathbf{v}_a(t)\, \delta(\mathbf{r} - \mathbf{r}_a(t)), \tag{5}$$
where $\mathbf{r}_a(t)$ is the position vector and
$$\mathbf{v}_a(t) = \frac{d\mathbf{r}_a}{dt} \tag{6}$$
is the velocity of the charged particle. Verify the statement of conservation of charge,
$$\frac{\partial}{\partial t}\rho(\mathbf{r}, t) + \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r}, t) = 0. \tag{7}$$

5. Show that the potential for a point charge, in three spatial dimensions,

$$\phi(\mathbf{r}) = \frac{q_a}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_a|}, \tag{8}$$

satisfies the differential equation

$$-\varepsilon_0 \nabla^2 \phi(\mathbf{r}) = q_a \delta^{(3)}(\mathbf{r}-\mathbf{r}_a). \tag{9}$$

Solve the corresponding differential equation in one spatial dimension,

$$-\varepsilon_0 \frac{d^2}{dx^2}\phi(x) = q_a \delta(x - x_a). \tag{10}$$

Hints:

(a) Using the definition of δ-function observe that

$$-\varepsilon_0 \frac{d^2}{dx^2}\phi(x) = 0, \quad \text{for} \quad x \neq x_a. \tag{11}$$

(b) Solve the homogeneous differential equation in Eq. (11) in terms of two integral constants in each of two regions,

$$\phi(x) = \begin{cases} a_1 x + b_1, & x < x_a, \\ a_2 x + b_2, & x > x_a. \end{cases} \tag{12}$$

(c) Integrate Eq. (10) from $x = x_a - \delta$ to $x = x_a + \delta$, for infinitesimal $\delta > 0$, to derive the boundary condition on

$$\frac{d}{dx}\phi(x). \tag{13}$$

(d) Argue that, for consistency, we also require the boundary condition

$$\phi(x_a - \delta) = \phi(x_a + \delta). \tag{14}$$

(e) Use the boundary conditions to determine two of the four integral constants in Eq. (12). In particular find $a_2 - a_1$ and $b_2 - b_1$. The solutions can be expressed in the form

$$\phi(x) = -\frac{q}{2\varepsilon_0}|x - x_a| + ax + b, \tag{15}$$

where $2a = a_1 + a_2$ and $2b = b_1 + b_2$.

6. (Ref. Milton's lecture notes.) A plane wave is described by electric and magnetic fields of the form

$$\mathbf{E} = \mathbf{e}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{16}$$

$$\mathbf{B} = \mathbf{b}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{17}$$

where $\mathbf{e}_0$ and $\mathbf{b}_0$ are constants. From Maxwell's equations in free space (no charges or currents)

(a) Determine the relation between $\mathbf{e}_0$, $\mathbf{b}_0$, and $\mathbf{k}$.

(b) Determine the relation between ω and $\mathbf{k}$.

(c) Verify the statement of conservation of energy for a plane wave.

(d) Verify the statement of conservation of momentum for a plane wave.

7. Problem 6.11, Jackson 3rd edition.