# Homework No. 07 (Fall 2013)

## PHYS 520A: Electromagnetic Theory I

Due date: Monday, 2013 Dec 2, 4.30pm

1. Consider the integral equation

$$K(t',t'') + i\int_0^t d\tau \Big[1 + it_<(t',\tau)\Big] K(\tau,t'') = \delta(t'-t''), \qquad 0 \leq \{t',t''\} \leq t, \tag{1}$$

where $t_<(t',\tau)$ stands for minimum of $t'$ and $\tau$.

(a) By differentiating the above integral equation in Eq. (1) twice with respect to $t'$ obtain the differential equation satisfied by $K(t',t'')$:

$$\left[\frac{\partial^2}{\partial t'^2} + 1\right] K(t',t'') = \frac{\partial^2}{\partial t'^2}\delta(t'-t''). \tag{2}$$

(b) Deduce the boundary conditions on $K(t',t'')$ from Eq. (1):

$$K(0,t'') = -i\int_0^t d\tau K(\tau,t''), \tag{3a}$$

$$K(t,t'') = K(0,t'') + \int_0^t d\tau\, \tau K(\tau,t''). \tag{3b}$$

Hint: Presume that the $\delta$-function in Eq. (1) does not contribute at $t'=0$ and $t'=t$. This assumption does not effect the solution, but leads to non-trivial contributions at the boundaries of integrals involving $K(t,t'')$.

(c) In terms of a Green's function $M(t',t'')$, which satisfies

$$\left[\frac{\partial^2}{\partial t'^2} + 1\right] M(t',t'') = \delta(t'-t''), \tag{4}$$

write

$$K(t',t'') = \frac{\partial^2}{\partial t'^2} M(t',t'') = \delta(t'-t'') - M(t',t''). \tag{5}$$

(d) Derive the continuity conditions for $M(t',t'')$, which are dictated by Eq. (4), to be

$$\{M(t',t'')\}_{t'=t''+\delta} - \{M(t',t'')\}_{t'=t''-\delta} = 0, \tag{6a}$$

$$\left\{\frac{\partial}{\partial t'} M(t',t'')\right\}_{t'=t''+\delta} - \left\{\frac{\partial}{\partial t'} M(t',t'')\right\}_{t'=t''-\delta} = 1, \tag{6b}$$

Additionally, the boundary conditions on $M(t',t'')$ are prescribed by the boundary conditions on $K(t',t'')$ in Eqs. (3a) and (3b).

(e) Write the solution to $M(t', t'')$ in the form

$$M(t', t'') = \begin{cases} \alpha(t'') \sin t' + \beta(t'') \cos t', & 0 \le t' < t'' \le t, \\ \eta(t'') \sin t' + \xi(t'') \cos t', & 0 \le t'' < t' \le t, \end{cases} \tag{7}$$

in terms of four arbitrary constants. Use the continuity conditions (6) to determine two of the four constants to obtain

$$K(t', t'') = \delta(t' - t'') - \alpha(t'') \sin t' - \xi(t'') \cos t' - \sin t_> \cos t_<, \tag{8}$$

where we have suppressed the $t'$ and $t''$ dependence in $t_<(t', t'')$ and $t_>(t', t'')$.

(f) Use the expression for $K(t', t'')$ in Eq. (8) into Eqs. (3a) and (3b) to obtain the equations determining $\alpha(t'')$ and $\xi(t'')$ to be

$$\alpha(t'') i[1 - \cos t] + \xi(t'')[1 + i \sin t] = i \cos t \cos t'' - \sin t'', \tag{9a}$$

$$\alpha(t'') \cos t - \xi(t'') \sin t = - \cos t \cos t'', \tag{9b}$$

and further obtain

$$\alpha(t'') = -e^{-it} \cos(t - t''), \tag{10a}$$

$$\xi(t'') = i e^{-i(t-t'')} \cos t. \tag{10b}$$

(g) Using Eqs. (10a) and (10b) in Eq. (8) obtain the solution to $K(t', t'')$ in the form

$$K(t', t'') = \delta(t' - t'') - i \cos(t - t') \cos(t - t'') - \sin(t - t_<) \cos(t - t_>). \tag{11}$$

(h) By substitution verify that Eq. (11) satisfies the original integral equation (1).