Exam No. 01 (Fall 2013)

PHYS 530B: Quantum Mechanics II

Date: 2013 Sep 26

1. An initial Stern-Gerlach measurement selects the + beam. A second + selection is made on this beam in a direction differing by angle θ. The probability amplitude for this outcome is
$$\langle\psi_+(0,0)|\psi_+(\theta,0)\rangle = \cos\frac{\theta}{2}, \tag{1}$$
and the probability of this outcome is
$$|\langle\psi_+(0,0)|\psi_+(\theta,0)\rangle|^2 = \cos^2\frac{\theta}{2}. \tag{2}$$
Determine the probability amplitude and probability of the outcome when the direction differs by angle $\theta + 2\pi$.

2. A quantum harmonic oscillator can be constructed out of two non-Hermitian operators, y and $y^\dagger$, that satisfy the commutation relation
$$[y, y^\dagger] = 1. \tag{3}$$
The eigenstate spectrum of the (Hermitian) number operator, $N = y^\dagger y$, represented by $|n\rangle$, where $n = 0, 1, 2, \ldots$, satisfy
$$N|n\rangle = n|n\rangle, \qquad y|n\rangle = \sqrt{n}|n-1\rangle, \qquad y^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{4}$$
   (a) Build the matrix representation of the lowering operator using
$$\langle n|y|n'\rangle = \begin{bmatrix} \langle 0|y|0\rangle & \langle 0|y|1\rangle & \langle 0|y|2\rangle & \langle 0|y|3\rangle & \langle 0|y|4\rangle & \cdots \\ \langle 1|y|0\rangle & \langle 1|y|1\rangle & \langle 1|y|2\rangle & \langle 1|y|3\rangle & \langle 1|y|4\rangle & \cdots \\ \langle 2|y|0\rangle & \langle 2|y|1\rangle & \langle 2|y|2\rangle & \langle 2|y|3\rangle & \langle 2|y|4\rangle & \cdots \\ \langle 3|y|0\rangle & \langle 3|y|1\rangle & \langle 3|y|2\rangle & \langle 3|y|3\rangle & \langle 3|y|4\rangle & \cdots \\ \langle 4|y|0\rangle & \langle 4|y|1\rangle & \langle 4|y|2\rangle & \langle 4|y|3\rangle & \langle 4|y|4\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5}$$
   Kindly calculate the first 5×5 block of the infinite dimensional matrix to report the pattern in the following questions.

   (b) Similarly, build the matrix representation of the raising operator $y^\dagger$.

   (c) Build the matrix representation of the number operator N.

(d) Using the constructions

$$y = \frac{1}{\sqrt{2\hbar}}(x + ip) \quad \text{and} \quad y^\dagger = \frac{1}{\sqrt{2\hbar}}(x - ip), \tag{6}$$

determine the matrix representations for the Hermitian operators, x and p. Check that x and p are Hermitian matrices.

(e) Determine the matrices for the operators xp and px, and verify the commutation relation

$$\frac{1}{i\hbar}[x, p] = 1. \tag{7}$$

3. A vector operator $\mathbf{V}$ is defined by the transformation property

$$\frac{1}{i\hbar}\left[\mathbf{V}, \delta\boldsymbol{\omega} \cdot \mathbf{J}\right] = \delta\boldsymbol{\omega} \times \mathbf{V}, \tag{8}$$

which states the commutation relations of components of $\mathbf{V}$ with those of angular momentum $\mathbf{J}$. Since a scalar operator S does not change under rotations it is defined by the corresponding transformation

$$\frac{1}{i\hbar}\left[S, \delta\boldsymbol{\omega} \cdot \mathbf{J}\right] = 0. \tag{9}$$

Given that the components of $\mathbf{V}_1$, and those of $\mathbf{V}_2 \times \mathbf{V}_3$, transform like a vector, show that the operator $\mathbf{V}_1 \cdot (\mathbf{V}_2 \times \mathbf{V}_3)$ transforms like a scalar operator.

4. Given that A and B are Hermitian operators, is the operator AB always Hermitian?