Homework No. 02 (Fall 2013)

PHYS 530B: Quantum Mechanics II

Due date: Wednesday, 2013 Sep 11, 4.30pm

1. A unitary matrix is defined by
$$U^\dagger U = 1, \tag{1}$$
where † stands for transpose and complex conjugation. Show that
$$U = e^{iH} \tag{2}$$
is unitary if H is hermitian,
$$H^\dagger = H. \tag{3}$$

2. Show that the combination $A^\dagger A$ is hermitian, irrespective of A being hermitian. Use this to deduce that the eigenvalues of $A^\dagger A$ is non-negative.

3. Prove that hermitian operators have real eigenvalues. Further, show that any two eigenfunctions belonging to distinct (unequal) eigenvalues of a hermitian operator are mutually orthogonal.

4. (Ref. Milton's notes.)

   (a) Consider three numerical vectors, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Show that
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0. \tag{4}$$

   (b) Now consider operators A, B, C. Show that
$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{5}$$

   (c) The multiplication property of the Pauli spin matrices can be written as
$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}). \tag{6}$$
   From this, show that
$$\frac{1}{i\hbar} \left[\frac{\hbar}{2}\boldsymbol{\sigma} \cdot \mathbf{a}, \frac{\hbar}{2}\boldsymbol{\sigma} \cdot \mathbf{b} \right] = \frac{\hbar}{2}\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}). \tag{7}$$

   (d) More generally, what is
$$\frac{1}{i\hbar} [\mathbf{J} \cdot \mathbf{a}, \mathbf{J} \cdot \mathbf{b}]? \tag{8}$$

(e) Use

$$A = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{a}, \qquad B = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{b}, \qquad \text{and} \qquad C = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{c} \tag{9}$$

in the result of problem (4b) to derive the result of problem (4a).

5. (Ref. Milton's notes.) A vector operator $\mathbf{V}$ is defined by the transformation property

$$\frac{1}{i\hbar}\left[\mathbf{V}, \delta\boldsymbol{\omega} \cdot \mathbf{J}\right] = \delta\boldsymbol{\omega} \times \mathbf{V}, \tag{10}$$

which states the commutation relations of components of $\mathbf{V}$ with those of angular momentum $\mathbf{J}$. Since a scalar operator S does not change under rotations it is defined by the corresponding transformation

$$\frac{1}{i\hbar}\left[S, \delta\boldsymbol{\omega} \cdot \mathbf{J}\right] = 0. \tag{11}$$

(a) Show that the scalar product of vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a scalar.

(b) Show that the vector product of vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a vector.