Homework No. 02A (BONUS) (Fall 2013)

PHYS 530B: Quantum Mechanics II

Due date: Monday, 2013 Sep 16, 4.30pm

1. We observed that the generator of a unitary transformation is arbitrary upto a phase $\delta\phi$. We shall determine the structure of the phase in this exercise. The generator is given by the expression
$$G = -H\delta t + \mathbf{P}\cdot\delta\boldsymbol{\varepsilon} + \mathbf{J}\cdot\delta\boldsymbol{\omega} + \mathbf{N}\cdot\delta\mathbf{v} + \hbar\delta\phi. \tag{1}$$
Our goal will be to construct the bilinear form $\delta_{[1,2]}\phi$.

   (a) Argue that $\delta_{[1,2]}\phi$ has to be antisymmetric in 12,
   $$\delta_{[1,2]}\phi = -\delta_{[2,1]}\phi. \tag{2}$$

   (b) Argue that $\delta_{[1,2]}$ has to be a scalar.

   (c) Show that the most general bilinear form for $\delta_{[1,2]}\phi$ is
   $$\delta_{[1,2]}\phi = K(\delta_1\boldsymbol{\omega}\cdot\delta_2\boldsymbol{\varepsilon} - \delta_2\boldsymbol{\omega}\cdot\delta_1\boldsymbol{\varepsilon}) + L(\delta_1\boldsymbol{\omega}\cdot\delta_2\mathbf{v} - \delta_2\boldsymbol{\omega}\cdot\delta_1\mathbf{v}) + M(\delta_1\boldsymbol{\varepsilon}\cdot\delta_2\mathbf{v} - \delta_2\boldsymbol{\varepsilon}\cdot\delta_1\mathbf{v}), \tag{3}$$
   where K, L, and M are constants.

   (d) The Jacobi identity, applied to three sets of infinitesimal transformations, implies that
   $$\begin{aligned} &K[\delta_{[1,2]}\boldsymbol{\omega}\cdot\delta_3\boldsymbol{\varepsilon} - \delta_3\boldsymbol{\omega}\cdot\delta_{[1,2]}\boldsymbol{\varepsilon} + \text{cycl. perm.}] \\ &+ L[\delta_{[1,2]}\boldsymbol{\omega}\cdot\delta_3\mathbf{v} - \delta_3\boldsymbol{\omega}\cdot\delta_{[1,2]}\mathbf{v} + \text{cycl. perm.}] \\ &+ M[\delta_{[1,2]}\boldsymbol{\varepsilon}\cdot\delta_3\mathbf{v} - \delta_3\boldsymbol{\varepsilon}\cdot\delta_{[1,2]}\mathbf{v} + \text{cycl. perm.}] = 0. \end{aligned} \tag{4}$$

      i. Verify that the coefficient of M vanishes identically.
      ii. Determine the simplified expression for the non-vanishing coefficient of K.
      iii. Determine the simplified expression for the non-vanishing coefficient of L.
      iv. Hence conclude that K and L must be zero.

   (e) Thus, conclude that
   $$\delta_{[1,2]}\phi = M(\delta_1\boldsymbol{\varepsilon}\cdot\delta_2\mathbf{v} - \delta_2\boldsymbol{\varepsilon}\cdot\delta_1\mathbf{v}). \tag{5}$$