Homework No. 04 (Fall 2013)

PHYS 530B: Quantum Mechanics II

Due date: Monday, 2013 Oct 7, 4.30pm

1. Using the properties of Pauli matrices show that
$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})^2 = 1. \tag{1}$$
Then, prove the identity
$$e^{i\frac{\theta}{2}(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})} = \cos\frac{\theta}{2} + i(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\sin\frac{\theta}{2}, \tag{2}$$
where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector constructed out of Pauli matrices, and $\hat{\mathbf{n}}$ is an arbitrary (numerical) unit vector. This represents a unitary transformation due to a rotation by angle θ about the axis $\hat{\mathbf{n}}$ for a spin-$\frac{1}{2}$ particle. Show that the wavefunctions differ by a sign under a rotation of $\theta = 2\pi$. But, how can physical measurements differ for rotations of $\theta = 2\pi$? Show that it does not, because physical quantities involve product of two wave functions.

2. Consider the construction
$$K(\lambda) = e^{-\lambda A} B e^{\lambda A} \tag{3}$$
in terms of two operators A and B. Show that
$$\frac{\partial K}{\partial \lambda} = [K, A]. \tag{4}$$
Evaluate the higher derivatives
$$\frac{\partial^n K}{\partial \lambda^n} \tag{5}$$
recursively. Thus, using Taylor expansion around $\lambda = 0$, show that
$$K(\lambda) = B + \lambda[B, A] + \frac{\lambda^2}{2!}[[B, A], A] + \frac{\lambda^3}{3!}[[[B, A], A], A] + \ldots. \tag{6}$$

3. Consider the following unitary transformations,
$$J_x(\phi) = e^{-\frac{i}{\hbar}\phi J_z} J_x e^{\frac{i}{\hbar}\phi J_z} \tag{7}$$
and
$$J_y(\phi) = e^{-\frac{i}{\hbar}\phi J_z} J_y e^{\frac{i}{\hbar}\phi J_z}. \tag{8}$$

By differentiating with respect to ϕ, and solving the resulting differential equations, derive

$$J_x(\phi) = J_x \cos\phi + J_y \sin\phi, \tag{9a}$$
$$J_y(\phi) = -J_x \sin\phi + J_y \cos\phi. \tag{9b}$$

Further, derive

$$J_+(\phi) = e^{-i\phi} J_+ \quad \text{and} \quad J_-(\phi) = e^{i\phi} J_-. \tag{10}$$

4. The transformation function relating the angular momentum eigenvectors between two coordinate frames, related by rotations described using Euler angles (ψ, θ, ϕ), is

$$\langle j, m | U(\psi, \theta, \phi) | j', m' \rangle = \delta_{jj'} e^{im\psi} U^{(j)}_{m,m'}(\theta) e^{im'\phi}, \tag{11}$$

where $U^{(j)}_{m,m'}(\theta)$ are generated by the relation

$$\frac{\bar{y}_+^{j+m}}{\sqrt{(j+m)!}} \frac{\bar{y}_-^{j-m}}{\sqrt{(j-m)!}} = \sum_{m'=-j}^{j} e^{im\psi} U^{(j)}_{m,m'}(\theta) e^{im'\phi} \frac{y_+^{j+m'}}{\sqrt{(j+m')!}} \frac{y_-^{j-m'}}{\sqrt{(j-m')!}}, \tag{12}$$

where

$$\begin{bmatrix} \bar{y}_+ \\ \bar{y}_- \end{bmatrix} = \begin{bmatrix} e^{i\frac{\psi}{2}} \cos\frac{\theta}{2} e^{i\frac{\phi}{2}} & e^{i\frac{\psi}{2}} \sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ -e^{-i\frac{\psi}{2}} \sin\frac{\theta}{2} e^{i\frac{\phi}{2}} & e^{-i\frac{\psi}{2}} \cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \end{bmatrix} \begin{bmatrix} y_+ \\ y_- \end{bmatrix}. \tag{13}$$

The above transformation function gives the probability amplitude relating measurements of angular momentum, or magnetic dipole moment, in two different directions related by the Euler angles.

(a) Extract the probability amplitudes relating measurements of angular momentum for $j = \frac{3}{2}$.

(b) Extract the probabilities

$$p(m, m'; \theta) = |\langle j, m | U(\psi, \theta, \phi) | j', m' \rangle|^2 \tag{14}$$

relating measurements of angular momentum for $j = \frac{3}{2}$.