Homework No. 05 (Fall 2013)

PHYS 530B: Quantum Mechanics II

Due date: Friday, 2013 Oct 18, 4.30pm

1. Show that the null-vector

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+) \tag{1}$$

can be expressed in terms of Pauli matrices in matrix form:

$$\mathbf{a} = y^T \sigma_y \frac{1}{2i} \boldsymbol{\sigma} y, \tag{2}$$

where $y^T = (y_+, y_-)$.

2. The generating function for the spherical harmonics, $Y_{lm}(\theta, \phi)$, is

$$\frac{1}{l!}\left(\mathbf{a} \cdot \frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\theta, \phi)\, \psi_{lm}, \tag{3}$$

where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{4}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \tag{5}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{6}$$

and

$$Y_{lm}(\theta, \phi) = e^{im\phi} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!}{(l-m)!}} \frac{1}{(\sin\theta)^m} \left(\frac{d}{d\cos\theta}\right)^{l-m} \frac{(\cos^2\theta - 1)^l}{2^l l!}. \tag{7}$$

Show that

$$\left(\mathbf{a} \cdot \frac{\mathbf{r}}{r}\right) \tag{8}$$

is unchanged by the substitution

$$y_+ \leftrightarrow y_-, \quad \theta \to -\theta, \quad \phi \to -\phi. \tag{9}$$

Thus, show that

$$Y_{lm}(\theta, \phi) = Y_{l,-m}(-\theta, -\phi). \tag{10}$$

3. Write down the explicit forms of the spherical harmonics $Y_{lm}(\theta, \phi)$ for $l = 0, 1, 2$, by completing the $l - m$ differentiations in Eq. (7). Use the result in Eq. (10) to reduce the work by about half.

4. Legendre polynomials of order l is given by (for $|t| < 1$)

$$P_l(t) = \left(\frac{d}{dt}\right)^l \frac{(t^2 - 1)^l}{2^l l!}. \tag{11}$$

   (a) Write down the explicit forms of the Legendre polynomials $P_l(t)$ for $l = 0, 1, 2, 3$, by completing the l differentiations in Eq. (11).

   (b) Show that the spherical harmonics for $m = 0$ involves the Legendre polynomials,

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi}} P_l(\cos\theta). \tag{12}$$

   (c) Using the orthonormality condition for the spherical harmonics

$$\int d\Omega\, Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'}\delta_{mm'} \tag{13}$$

   recognize the orthogonality statement for Legendre polynomials,

$$\frac{1}{2}\int_{-1}^{1} dt\, P_l(t) P_{l'}(t) = \frac{\delta_{ll'}}{2l + 1}. \tag{14}$$

   Use

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \tag{15}$$

   to check this explicitly for $l, l' = 0, 1, 2$.

5. An example of a null-vector is

$$\mathbf{a} = (-i\cos\alpha, -i\sin\alpha, 1). \tag{16}$$

   Identify the corresponding $y_\pm$ in Eq. (1) to show that, now, ψ_{lm} in Eq. (3) is

$$\psi_{lm} = \frac{e^{-im\left(\alpha - \frac{\pi}{2}\right)}}{\sqrt{(l + m)!(l - m)!}}. \tag{17}$$

   Then, integrate Eq. (3) to derive an integral representation for spherical harmonics,

$$\frac{1}{l!}\int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{im\alpha}[\cos\theta - i\sin\theta\cos(\phi - \alpha)]^l = \sqrt{\frac{4\pi}{2l + 1}} \frac{i^m\, Y_{lm}(\theta, \phi)}{\sqrt{(l + m)!(l - m)!}}. \tag{18}$$

   By setting $m = 0$ derive the corresponding integral representation for Legendre polynomial $P_l(\cos\theta)$.