Homework No. 07 (Fall 2013)

PHYS 530B: Quantum Mechanics II

Due date: Wednesday, 2013 Nov 13, 4.30pm

1. The polar equation of a conic of eccentricity ε is

$$r = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta}, \tag{1}$$

where $2a$ is the major-axis of the ellipse. The directrix of an ellipse is a line perpendicular to the major-axis at a distance d from the focus (origin). For an ellipse, the ratio between the radial distance of a point on the ellipse from the origin, and the distance of the point from the directrix, is the eccentricity. Thus, determine d in terms of a and ε.

2. The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Verify the following:

   (a) $\mathbf{r}\times\mathbf{p} + \mathbf{p}\times\mathbf{r} = 0$.

   (b) $\mathbf{r}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{r} = 3i\hbar$.

   (c) $(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{p}) - (\mathbf{a}\cdot\mathbf{p})(\mathbf{b}\cdot\mathbf{r}) = i\hbar(\mathbf{a}\cdot\mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ and numerical.

   (d) $\mathbf{r}\times(\mathbf{r}\times\mathbf{p}) = \mathbf{r}\,\mathbf{p}\cdot\mathbf{r} - \mathbf{p}r^2 + i\hbar\mathbf{r}$.

3. Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the following:

   (a) $\mathbf{L}\times\mathbf{L} = i\hbar\mathbf{L}$.

   (b) $\mathbf{p}\times\mathbf{L} + \mathbf{L}\times\mathbf{p} = 2i\hbar\mathbf{p}$.

   (c) $-\mathbf{L}\times\mathbf{p}\cdot\frac{\mathbf{r}}{r} = L^2\frac{1}{r} = \frac{1}{r}L^2$.

4. Verify the following equation of motion for the hydrogen atom,

$$\frac{1}{i\hbar}[\mathbf{p}, H] = -\frac{Ze^2\mathbf{r}}{r^3}, \tag{2}$$

where the Hamiltonian is

$$H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}. \tag{3}$$

5. Verify that the axial vector,

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Ze^2}\frac{1}{2}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}), \tag{4}$$

satisfies

$$\mathbf{A}\cdot\mathbf{L} = 0 \qquad \text{and} \qquad \mathbf{L}\cdot\mathbf{A} = 0. \tag{5}$$