Homework No. 09 (Fall 2013)

PHYS 530B: Quantum Mechanics II

Due date: No submission required

1. Verify, by substitution, that
$$G_0(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi}\left[A\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} + B\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right] \tag{1}$$
with the constraint $A + B = 1$ is a particular solution to the Green's function equation
$$\left[\nabla^2 + k^2\right] G_0(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{2}$$
Hint: Prove that
$$-\nabla^2 \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{3}$$

2. Consider the Schrödinger equation
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t)\right]\psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t). \tag{4}$$
   (a) Derive the statement of probability conservation (continuity equation),
   $$\frac{\partial}{\partial t}\rho(\mathbf{r}, t) + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r}, t) = s(\mathbf{r}, t), \tag{5}$$
   where
   $$\rho(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2, \tag{6}$$
   $$\mathbf{j}(\mathbf{r}, t) = \frac{\hbar}{2im}\left[\psi^*(\mathbf{r}, t)\boldsymbol{\nabla}\psi(\mathbf{r}, t) - \psi(\mathbf{r}, t)\boldsymbol{\nabla}\psi^*(\mathbf{r}, t)\right], \tag{7}$$
   $$s(\mathbf{r}, t) = \frac{2}{\hbar}\left[\text{Im}\, V(\mathbf{r}, t)\right]|\psi(\mathbf{r}, t)|^2. \tag{8}$$
   (b) For a time-independent potential the scattering wavefunction can be expressed in the form, after replacing $\psi(\mathbf{r}, t) \to e^{-iEt/\hbar}\psi(\mathbf{r})$,
   $$\psi_{r\to\infty}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta, \phi)\frac{e^{ikr}}{r}. \tag{9}$$
   Construct the continuity equation for the above scattering wavefuntion. Then, after integrating over a large sphere ($r \to \infty$), derive
   $$\lim_{r\to\infty}\oint d\mathbf{S}\cdot\mathbf{j}(\mathbf{r}) = \lim_{r\to\infty}\int d^3r\, s(\mathbf{r}). \tag{10}$$

(c) Show that

$$\lim_{r\to\infty}\oint d\mathbf{S}\cdot\mathbf{j}(\mathbf{r}) = \frac{\hbar k}{m}\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, |f(\theta,\phi)|^2 - \frac{4\pi\hbar}{m}\left[\text{Im}\, f(0)\right]. \tag{11}$$

(d) Thus, derive the optical theorem,

$$\sigma_{\text{scatt.}} + \sigma_{\text{abs.}} = \frac{4\pi}{k}\left[\text{Im}\, f(0)\right], \tag{12}$$

where

$$\sigma_{\text{scatt.}} = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, |f(\theta,\phi)|^2, \tag{13}$$

$$\sigma_{\text{abs.}} = -\lim_{r\to\infty}\frac{2m}{\hbar^2 k}\int d^3r\, \left[\text{Im}\, V(\mathbf{r})\right] |\psi(\mathbf{r})|^2. \tag{14}$$

3. For potentials that are independent of ϕ the axial symmetry allows us to write the leading order contribution to scattering amplitude in the eikonal approximation (small angle large momentum) as

$$f^{(0)}(\theta) = \frac{k}{i}\int_0^\infty bdb\, J_0(kb\theta)\left[e^{i\chi(b)} - 1\right], \tag{15}$$

where

$$\chi(b) = -\frac{k}{2E}\int_{-\infty}^\infty dz\, V(b,z). \tag{16}$$

(a) For potentials that have no imaginary parts show that the optcal theorem takes the form

$$\sigma_{\text{scatt.}} = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, |f(\theta,\phi)|^2 = \frac{4\pi}{k}\left[\text{Im}\, f(0)\right]. \tag{17}$$

(b) Using the second equality in Eq. (17) verify that

$$\sigma_{\text{scatt.}} = 8\pi\int_0^\infty bdb\, \sin^2\left[\frac{\chi(b)}{2}\right]. \tag{18}$$

(c) Using the first equality in Eq. (17) and arguing that small angle approximation is represented by the range of integration $0 \le \theta < V/E$ show that

$$\sigma_{\text{scatt.}} = 2\pi k^2\int_0^\infty bdb\int_0^\infty b'db'\left[e^{i\chi(b)} - 1\right]\left[e^{-i\chi(b')} - 1\right]\int_0^{\frac{V}{E}} \theta d\theta J_0(kb\theta)J_0(kb'\theta). \tag{19}$$

(d) Show that in the limit

$$ka\frac{V}{E} \to \infty \tag{20}$$

the optical theorem, which is the statement of probability conservation, is satisfied. Hint: Use

$$\int_0^\infty xdx J_0(tx)J_0(t'x) = \frac{\delta(x-x')}{x}. \tag{21}$$