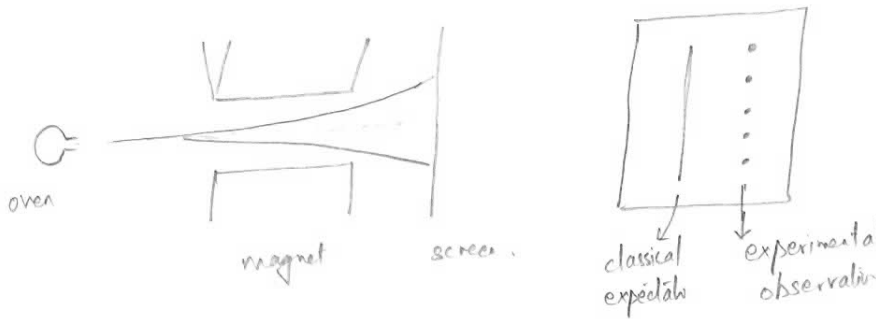
SET - 02

1 Stern - Gerlach experiment



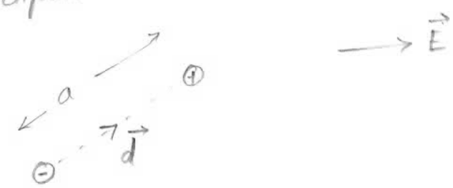

oven - source of atoms (neutral) with magnetic dipole moment.
eg. Silver atoms

magnet -


- knife edge creates an inhomogeneous magnetic field.

screen - a continuous distribution was expected using classical analysis. But, for spin-$\frac{1}{2}$ silver atoms only two spots were experimentally observed.

(2) Electric dipole moment




$\vec{d} = q\vec{a}$ — dipole moment

(3) Force

$\vec{r}_+ = \vec{r}_- + \vec{a}$

let $\vec{r}_- = \vec{r}$

$$
\begin{aligned}
\vec{F} &= q\,\vec{E}(\vec{r}_+) - q\,\vec{E}(\vec{r}_-) \\
&= q\,\vec{E}(\vec{r}+\vec{a}) - q\,\vec{E}(\vec{r}) \\
&= q\,\vec{a}\cdot\vec{\nabla}\,\vec{E}(\vec{r}) \\
&= \vec{d}\cdot\vec{\nabla}\,\vec{E}(\vec{r}) \\
&= (\vec{d}\cdot\vec{\nabla})\,\vec{\nabla}\,\phi(\vec{r}) \\
&= \vec{\nabla}\,(\vec{d}\cdot\vec{\nabla})\,\phi(\vec{r}) \\
&= \vec{\nabla}\;\vec{d}\cdot\vec{E}(\vec{r})
\end{aligned}
$$

— ($\vec{E} = -\vec{\nabla}\phi$ for electrostatics)

($\vec{\nabla}\,(\vec{d}\cdot\vec{\nabla}) = \vec{d}\cdot\vec{\nabla}\,\vec{\nabla}$ since $\vec{d}$ is a property of the atom, no $\vec{r}$ dependence)

(4) Comparing with

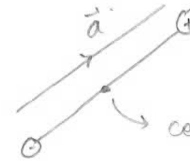
$$\vec{F} = -\vec{\nabla}\,\varepsilon$$

$\varepsilon$ — energy

we identify

$$\varepsilon = -\vec{d}\cdot\vec{E}(\vec{r})$$

(5) Torque on an electric dipole moment. (about center of mass)




$$\vec{\tau} = \sum_{a=1}^{2} \vec{r}_a \times \vec{F}_a$$

$$= \frac{\vec{a}}{2} \times q\,\vec{E}(\vec{r}_+) + \frac{\vec{a}}{2} \times q\,\vec{E}(\vec{r}_-)$$

$$= q\frac{\vec{a}}{2} \times \vec{E}(\vec{r}_- + \vec{a}) + q\frac{\vec{a}}{2} \times \vec{E}(\vec{r}_-)$$

$$\approx q\,\vec{a} \times \vec{E}(\vec{r}_-)$$

$$= \vec{d} \times \vec{E}(\vec{r})$$

(6) Summary

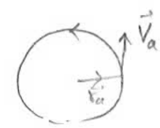
$$\vec{d} = q\,\vec{a} \qquad \vec{F} = \vec{\nabla}(\vec{d}\cdot\vec{E})$$

$$\varepsilon = \vec{d}\cdot\vec{E} \qquad \vec{\tau} = \vec{d} \times \vec{E}$$

(7) Magnetic dipole moment

$$\vec{\mu} = \frac{q}{2c}\,\vec{r}_a \times \vec{V}_a$$

(8) Consider a charge moving in a circle




$\vec{r}_a \times \vec{V}_a = vr\,\hat{n}$

$$\vec{\mu} = \hat{n}\,\frac{1}{c}\,\frac{qvr}{2}$$

$$= \hat{n}\,\frac{1}{c}\,\frac{qv}{2\pi r}\,\pi r^2$$

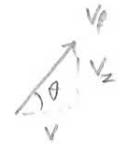
$$= \hat{n}\,\frac{1}{c}\,\underbrace{I}_{\text{current}}\,\underbrace{A}_{\text{area}}$$

$$I = \frac{qn}{\Delta t} = \frac{qv}{2\pi r}$$

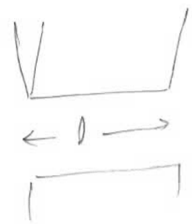
$n$ = number of times it circles

$$n = \frac{v\,\Delta t}{2\pi r}$$

(12)


$$\tan\theta = \frac{v_z}{v} = \frac{p_z}{p}$$




$$F = \frac{dp_z}{dt} = \mu_z \frac{\partial H_z}{\partial z}$$

$$p_z = \Delta t \; \mu_z \frac{\partial H_z}{\partial z}$$

$v \Delta t = l.$

$$= \frac{l}{v} \mu_z \frac{\partial H_z}{\partial z}$$

$$\tan\theta = \frac{l}{mv^2} \mu_z \frac{\partial H_z}{\partial z}$$

(13) Estimation of $\theta$.

$\mu_z \sim 10^{-27} \frac{J}{\text{Gauss}}$

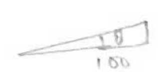
$\frac{\partial H_z}{\partial z} \sim 10^4 \frac{G}{cm} = 10^6 \frac{G}{m}$

$l = 10\,cm = 10^{-1}\,m$

$\frac{1}{2} mv^2 = kT \quad (T = 10^3\,K)$

$mv^2 = 2kT$
$= 2 \times (1.4 \times 10^{-23})\, 10^3$
$\sim 3 \times 10^{-20}\,J$

$$\tan\theta = \frac{(10^{-1} m)\left(10^{-27} \frac{J}{G}\right)\left(10^6 \frac{G}{m}\right)}{3 \times 10^{-20}\,J} \sim 10^{-2} = \frac{1}{100}$$




$\theta \sim 10^{-2}\,\text{rad}\; \frac{180\,\text{deg}}{\pi\,\text{rad}} \sim \frac{1}{2}$ degree.

$\tan\theta = \frac{1}{100} \rightarrow$ 1 cm deflection on a screen 1 meter away.

(14) Deflection :


$$\Delta z = \frac{1}{2} a \Delta t^2$$

$$= \frac{1}{2} \frac{1}{m} \mu_z \frac{\partial H_z}{\partial z} \left(\frac{l}{v}\right)^2.$$

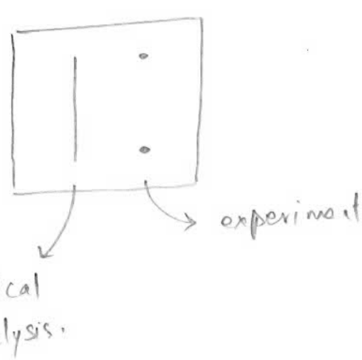
deflection is linear in $\mu_z$

(15) $\mu_z = \mu \cos\theta$.

→ $\theta$ is randomly distributed due to the thermal energy inside the oven.

→ thus, $\mu_z = \mu \cos\theta$ can take a continuous set of values between $+\mu$ and $-\mu$.

→ $\Delta z \propto \mu_z$ thus, the atoms are expected to deflect continuously

→ 

(16) <u>Pauli matrices</u>

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$[\sigma_i , \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = 2 i \epsilon_{ijk} \sigma_k$$

$$\{\sigma_i , \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij}$$

(17) $\mathrm{Tr}\,(\sigma_i) = 0$,

$\det\,(\sigma_i) = -1$,

which together implies the eigen values are $\pm 1$.

(18) Using (16), adding 2nd & 3rd eq.,

$$\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k$$

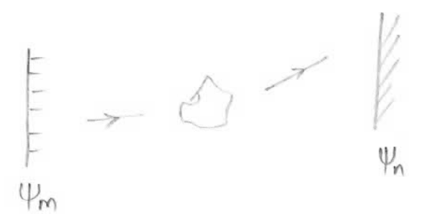
which can be expressed as

$$\vec{\sigma} \cdot \vec{A} \; \vec{\sigma} \cdot \vec{B} = \vec{A} \cdot \vec{B} + i \, \vec{\sigma} \cdot \vec{A} \times \vec{B}$$

(19) Consider a scattering problem.

$$H = H_0 + H_{int}$$

Eigenfunctions are basis set for wave function




Scattering amplitude $= \langle \Psi_m | \Psi_n \rangle$

Probability amplitude $= | \langle \Psi_m | \Psi_n \rangle |^2$

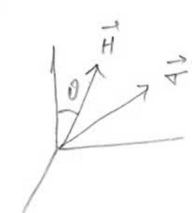
(20) Interaction energy in Stern-Gerlach experiment is

$$\begin{aligned} H_{int} &= -\vec{\mu} \cdot \vec{H} \\ &= -\Gamma\, \vec{J} \cdot \vec{H} \\ &= -\frac{\Gamma}{2}\, \vec{\sigma} \cdot \vec{H} \\ &= -\frac{\Gamma}{2} H \left[ \sigma_x \sin\theta \cos\phi + \sigma_y \sin\theta \sin\phi + \sigma_z \cos\theta \right] \end{aligned}$$

$\vec{J} = \frac{1}{2} \vec{\sigma}$ for spin-$\frac{1}{2}$.

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

$\vec{H} = H(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$




$$H_{int} = -\frac{\Gamma}{2} H \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix}$$

(21) Eigen values

$$\det \begin{pmatrix} \cos\theta - \lambda & \sin\theta \, e^{-i\phi} \\ \sin\theta \, e^{i\phi} & -\cos\theta - \lambda \end{pmatrix} = 0$$

$$(\lambda - \cos\theta)(\lambda + \cos\theta) - \sin^2\theta = 0$$

$$\lambda^2 - \cos^2\theta - \sin^2\theta = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

(22) $\underline{\underline{\lambda = +1}}$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \, e^{-i\phi} \\ \sin\theta \, e^{i\phi} & -\cos\theta - 1 \end{pmatrix} \psi_+(\theta, \phi) = 0$$

$$\psi_+(\theta, \phi) = A \begin{pmatrix} \cos\frac{\theta}{2} \, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2} \, e^{i\frac{\phi}{2}} \end{pmatrix}$$

Normalization implies $A^2 = 1 \Rightarrow A = e^{i\alpha}$

$$\psi_+(\theta, \phi) = e^{i\alpha} \begin{pmatrix} \cos\frac{\theta}{2} \, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2} \, e^{i\frac{\phi}{2}} \end{pmatrix}$$

(23) $\underline{\underline{\lambda = -1}}$

$$\psi_-(\theta, \phi) = e^{i\beta} \begin{pmatrix} -\sin\frac{\theta}{2} \, e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2} \, e^{i\frac{\phi}{2}} \end{pmatrix}$$

(24) Orthonormality

$$\psi_+^{\dagger}(\theta,\phi)\ \psi_+(\theta,\phi) = 1$$

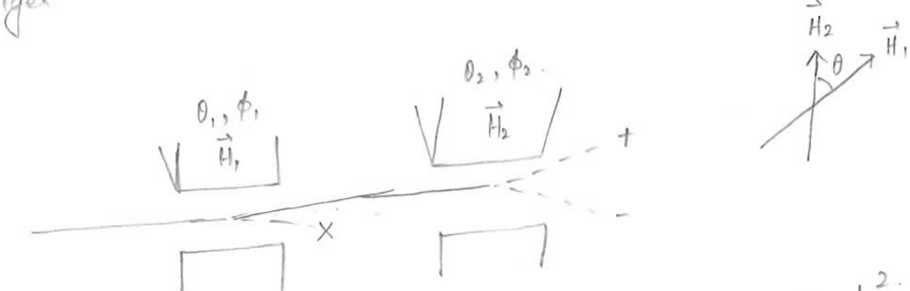
$$\psi_-^{\dagger}(\theta,\phi)\ \psi_-(\theta,\phi) = 1$$

$$\psi_+^{\dagger}(\theta,\phi)\ \psi_-(\theta,\phi) = 0$$

$$\psi_-^{\dagger}(\theta,\phi)\ \psi_+(\theta,\phi) = 0$$

Thus, $\psi_\pm(\theta,\phi)$ form a complete orthonormal eigen vectors for $H_{int}$.

(25)




$$P\left([+;0,0] \longrightarrow [+;\theta,0]\right) = \left|\left\langle \psi_+(0,0) \middle| \psi_+(\theta,0)\right\rangle\right|^2$$

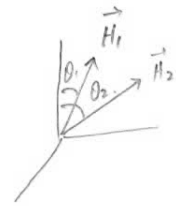
$$= \left|\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}\right|^2$$

$$= \cos^2\frac{\theta}{2}$$

$$P\left([+;0,0] \longrightarrow [-;\theta,0]\right) = \left|\left\langle \psi_+(0,0) \middle| \psi_-(\theta,0)\right\rangle\right|^2$$

$$= \left|\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}\right|^2$$

$$= \sin^2\frac{\theta}{2}$$




(26) $$P\left([+;\theta_1,\phi_1] \longrightarrow [+;\theta_2,\phi_2]\right)$$

$$= \left| \langle \psi_+(\theta_1,\phi_1) \,|\, \psi_+(\theta_2,\phi_2) \rangle \right|^2$$

$$= \left| \begin{pmatrix} \cos\frac{\theta_1}{2}\, e^{i\frac{\phi}{2}} & \sin\frac{\theta_1}{2}\, e^{-i\frac{\phi_1}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_2}{2}\, e^{-i\frac{\phi_2}{2}} \\ \sin\frac{\theta_2}{2}\, e^{i\frac{\phi_2}{2}} \end{pmatrix} \right|^2$$

$$= \left| \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\, e^{i\frac{\phi_1-\phi_2}{2}} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\, e^{-i\frac{\phi_1-\phi_2}{2}} \right|^2$$

$$\downarrow$$

$$= \frac{1+\cos\Theta}{2}$$

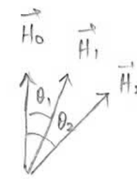
where $\quad \cos\Theta = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$

(27) Similarly show

$$P\left([+;\theta_1,\phi_1] \rightarrow [-;\theta_2,\phi_2]\right) = \frac{1-\cos\Theta}{2}$$

(28)
$$p\left([+;\theta_1,0] \to [+;\theta_2,0]\right) = \cos^2 \frac{\theta_1-\theta_2}{2}$$
$$p\left([+;\theta_1,0] \to [-;\theta_2,0]\right) = \sin^2 \frac{\theta_1-\theta_2}{2}$$

(29)
$$p\left([+;0,0] \to [+;\tfrac{\pi}{2},0]\right) = \frac{1}{2}$$
$$p\left([+;0,0] \to [-;\tfrac{\pi}{2},0]\right) = \frac{1}{2}.$$




(30)
$$p\left([+;0] \to [+;\theta_1] \to [+;\theta_2]\right)$$
$$= \left| \langle \psi_+(0) | \psi_+(\theta_1)\rangle \langle \psi_+(\theta_1) | \psi_+(\theta_2)\rangle \right|^2$$
$$= \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_1}{2} \\ \sin\frac{\theta_1}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_1}{2} & \sin\frac{\theta_1}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2} \end{pmatrix} \right|^2$$
$$= \left| \cos\frac{\theta_1}{2} \cos\frac{\theta_1-\theta_2}{2} \right|^2$$

(31) Similarly.
$$p\left([+;0] \to [-;\theta_1] \to [+;\theta_2]\right)$$
$$= \left| \sin\frac{\theta_1}{2} \sin\frac{\theta_1-\theta_2}{2} \right|^2$$

32 Destructive interference

$$p([+;0] \to [+;\pi]) = 0$$

$$p([+;0] \to [\pm;\theta] \to [+;\pi])$$

$$= \left| \langle \psi_+(0) | \psi_+(\theta) \rangle \langle \psi_+(\theta) | \psi_+(\pi) \rangle + \langle \psi_+(0) | \psi_-(\theta) \rangle \langle \psi_-(\theta) | \psi_+(\pi) \rangle \right|^2$$

$$= \left| \cos\frac{\theta}{2} \sin\frac{\theta}{2} - \sin\frac{\theta}{2} \cos\frac{\theta}{2} \right|^2$$

$$= 0$$

33 Constructive interference

$$p([+;0] \to [-;\pi]) = 1$$

$$p([+;0] \to [\pm;\theta] \to [-;\pi])$$

$$= \left| \langle \psi_+(0) | \psi_+(\theta) \rangle \langle \psi_+(\theta) | \psi_-(\pi) \rangle + \langle \psi_+(0) | \psi_-(\theta) \rangle \langle \psi_-(\theta) | \psi_-(\pi) \rangle \right|^2$$

$$= \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right|^2$$

$$= \left| \cos\frac{\theta}{2} \left(-\cos\frac{\theta}{2}\right) + \left(-\sin\frac{\theta}{2}\right) \sin\frac{\theta}{2} \right|^2$$

$$= 1$$

34 $N$ successive (infinitesimal) Stern-Gerlach rotations

$$P\left([+;0] \to [+;\theta_1] \to [+;\theta_2] \to \dots \to [+;\theta_n]\right)$$

$$= \left| \cos\frac{\theta_1}{2} \cos\frac{\theta_1-\theta_2}{2} \cos\frac{\theta_2-\theta_3}{2} \dots \cos\frac{\theta_{n-1}-\theta_n}{2} \right|^2$$

Let $\theta_1 = \theta_2 - \theta_1 = \theta_3 - \theta_2 = \dots = \theta_n - \theta_{n-1} = \epsilon$

$$P = \left[\cos\frac{\epsilon}{2}\right]^{2n}$$

Let $n\epsilon = \theta$ $\quad \epsilon \to 0,\ n \to \infty,\ \theta = \text{fixed}$

$$P = \left[\cos\frac{\epsilon}{2}\right]^{\frac{2\theta}{\epsilon}}$$

$$\approx \left(1 - \frac{\epsilon^2}{8}\right)^{\frac{2\theta}{\epsilon}}$$

$$\approx 1 - \frac{2\theta}{\epsilon}\frac{\epsilon^2}{8} + \frac{2\theta}{\epsilon}\left(\frac{2\theta}{\epsilon} - 1\right)\left(-\frac{\epsilon^2}{8}\right)^2 \frac{1}{2} + \dots$$

$$\approx 1 - \theta\frac{\epsilon}{4}$$

$$= 1 \quad \left(\lim_{\epsilon \to 0}\right)$$

Reason this.

(35) Heuristic reasoning $\qquad \vec{H} = H\hat{z}$

$$\vec{\tau} = \vec{\mu} \times \vec{H}$$

$$\frac{d\vec{J}}{dt} = r\,\vec{J} \times \vec{H} = rH\left[ J_y\,\hat{x} - J_x\,\hat{y} + 0\,\hat{z} \right]$$

$$\frac{dJ_z}{dt} = 0 \qquad \frac{dJ_x}{dt} = rH\,J_y$$

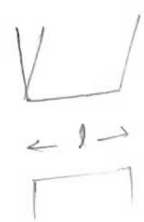
$$\frac{dJ_y}{dt} = -rH\,J_x$$

(36)
$$\frac{d^2}{dt^2} J_x = -r^2 H^2 J_x \qquad \omega = rH$$

$$J_x(t) = A\cos\omega t$$

$$J_y(t) = -A\sin\omega t$$

(37) Precession.




$$\Delta\phi = \omega\,\Delta t = rH\,\frac{l}{v}$$

38 How much $H$ is required to precess $\Delta\phi = 1$ radian $= 57$ deg.

$$\Delta\phi = \gamma H \frac{l}{v}.$$

$$H = \frac{v\,\Delta\phi}{\gamma\, l}.$$

$$= \frac{\left(10^2 \frac{m}{s}\right)\left(1 \text{ rad}\right)}{\left(10^7\ G^{-1} \frac{1}{s}\right)\left(10^{-2} m\right)}$$

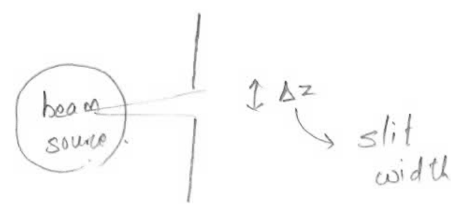
$$\sim 10^{-3} \text{ Gauss}$$

$$\gamma = \frac{\mu}{J} = \frac{10^{-27}\ \frac{J}{G}}{\hbar\ \ J\sec} = 10^{7}\ G^{-1}\sec^{-1}$$

Earth's $H = 1$ Gauss.

Very sensitive!

39 $$\Delta\phi = \gamma \frac{l}{v} \frac{\partial H}{\partial z} \Delta z$$




Since precession angle is very sensitive to the magnetic field in S-G experiment, the size of the width leads to a significant error in $\delta\phi$, which leads to loss of information in $J_x$ and $J_y$. Thus.

$$\langle J_x \rangle = 0$$
$$\langle J_y \rangle = 0$$