SET-03

Angular momentum generator

① Let us consider rotations in 3-d space, in which a general rotation can be written in terms of consecutive rotations about x, y, and z axis.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\omega_3 & \sin\omega_3 & 0 \\ -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 1 & 0 \\ \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

② For infinitesimal rotations we use

$$\cos\omega_i \sim 1$$
$$\sin\omega_i \sim \delta\omega_i$$

to obtain

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & \delta\omega_3 & 0 \\ -\delta\omega_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\delta\omega_2 \\ 0 & 1 & 0 \\ \delta\omega_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\omega_1 \\ 0 & -\delta\omega_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \delta\omega_3 & -\delta\omega_2 \\ -\delta\omega_3 & 1 & \delta\omega_1 \\ \delta\omega_2 & -\delta\omega_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which can be written as

$$\vec{x}' = \vec{x} - \delta\vec{\omega} \times \vec{x}$$

$$\delta\vec{\omega} \times \vec{x} = \begin{pmatrix} \delta\omega_2 x_3 - \delta\omega_3 x_2 \\ -[\delta\omega_1 x_3 - \delta\omega_3 x_1] \\ \delta\omega_1 x_2 - \delta\omega_2 x_1 \end{pmatrix}$$

3 Let us next consider rotations in a Hilbert space.

$$|\rangle' = U_{rot}^{-1}|\rangle$$
$$= e^{-\frac{i}{\hbar}G_{rot}}|\rangle$$

(since any unitary transformation can be written in terms of a Hermitian operator. $U = e^{\frac{i}{\hbar}G}$.)

4 For infinitesimal rotations we can write

$$|\rangle' = \left(1 - \frac{i}{\hbar} G_{rot}\right)|\rangle$$

5 Let the operators J_1, J_2, and J_3 be the generators of rotations due to $\delta\omega_1$, $\delta\omega_2$, and $\delta\omega_3$, respectively.

$$1 - \frac{i}{\hbar} G_{rot} = \left(1 - \frac{i}{\hbar} J_1\,\delta\omega_1\right)\left(1 - \frac{i}{\hbar} J_2\,\delta\omega_2\right)\left(1 - \frac{i}{\hbar} J_3\,\delta\omega_3\right)$$
$$= 1 - \frac{i}{\hbar}\vec{J}\cdot\delta\vec{\omega} + O(\delta\omega)^2$$

6 Thus, we have, using 3 to 5

$$|\rangle' = \left(1 - \frac{i}{\hbar}\vec{J}\cdot\delta\vec{\omega}\right)|\rangle$$

7 For the case of rotations in 3-D space we have the corresponding definition of angular momentum (generator of rotation)

$$\vec{x}' = \Big(\overleftrightarrow{I} - \frac{i}{\hbar}\underbrace{\overleftrightarrow{\vec{J}}\cdot\delta\vec{\omega}}_{\overleftrightarrow{G}}\Big)\cdot\vec{x}$$

$\rightarrow$ $\overleftrightarrow{\vec{J}}$ is a vector, each component of which is a matrix. Eg: Pauli matrix.

8 Comparing 2 and 7 we have.

$$\vec{x}' = \vec{x} - \delta\vec{\omega} \times \vec{x}$$

$$\vec{x}' = \Big(\overleftrightarrow{1} - \frac{i}{\hbar} \underbrace{\overleftrightarrow{\vec{J}} \cdot \delta\vec{\omega}}_{\overleftrightarrow{G}}\Big) \cdot \vec{x}$$

$$\Rightarrow \quad \frac{i}{\hbar} \overleftrightarrow{G} \cdot \vec{x} = \delta\vec{\omega} \times \vec{x}$$

$$G_{ik} x_k = -i\hbar \, \epsilon_{ijk} \, \delta\omega_j \, x_k.$$

$$G_{ik} = -i\hbar \, \epsilon_{imk} \, \delta\omega_m$$

G_{ik}: i, k → matrix index.

ϵ_{imk}: i, k → matrix index; m → vector index.

$\delta\omega_m$ → vector index.

9 Further

$$\left(\overleftrightarrow{\vec{J}} \cdot \delta\vec{\omega}\right)_{ik} = G_{ik} = -i\hbar \, \epsilon_{imk} \, \delta\omega_m$$

$$(J_m)_{ik} \, \delta\omega_m = -i\hbar \, \epsilon_{imk} \, \delta\omega_m$$

$$(J_m)_{ik} = -i\hbar \, \epsilon_{imk}$$

10 Using 9 the angular momentum is realized in 3-D space by the matrices.

$$\overleftrightarrow{J_1} = (J_1)_{ik} = -i\hbar\,\epsilon_{i1k} = -i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\overleftrightarrow{J_2} = (J_2)_{ik} = -i\hbar\,\epsilon_{i2k} = -i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\overleftrightarrow{J_3} = (J_3)_{ik} = -i\hbar\,\epsilon_{i3k} = -i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

11 In the Hilbert space consider the following sequence of operations

$$
\begin{tikzcd}
 & |\,\rangle_1 \arrow[r, "G_2"] & |\,\rangle_{21} \\
|\,\rangle_0 \arrow[ur, "G_1"] \arrow[dr, "G_2"'] & & \\
 & |\,\rangle_2 \arrow[r] & |\,\rangle_{12} \arrow[uu, "G_{[1,2]}"']
\end{tikzcd}
$$

12
$$
\begin{aligned}
|\,\rangle_{21} &= U_2^{-1}\,|\,\rangle_1 = U_2^{-1}U_1^{-1}\,|\,\rangle_0 \\
&= \left(1 - \frac{i}{\hbar}G_2\right)\left(1 - \frac{i}{\hbar}G_1\right)|\,\rangle_0 \\
&= \left(1 - \frac{i}{\hbar}G_2 - \frac{i}{\hbar}G_1 + \left(\frac{i}{\hbar}\right)^2 G_2G_1\right)|\,\rangle_0
\end{aligned}
$$

Similarly
$$
|\,\rangle_{12} = \left(1 - \frac{i}{\hbar}G_1 - \frac{i}{\hbar}G_2 + \left(\frac{i}{\hbar}\right)^2 G_1G_2\right)|\,\rangle_0
$$

$$[X, Y] = XY - YX$$

13 Using 12
$$
\begin{aligned}
|\,\rangle_{21} &= |\,\rangle_{12} - \left(\frac{i}{\hbar}\right)^2 [G_1, G_2]\,|\,\rangle_0 \\
&= \left(1 - \left(\frac{i}{\hbar}\right)^2 [G_1, G_2]\right)|\,\rangle_{12} \qquad \text{(to leading order)}
\end{aligned}
$$

14 Since $|\rangle_{21}$ and $|\rangle_{12}$ are member of the Hilbert space they are connected by an unitary transformation,

$$|\rangle_{21} = \left(1 + \frac{i}{\hbar} G_{[1,2]}\right) |\rangle_{12}$$

15 Comparing 13 and 14 we identify

$$-\left(\frac{i}{\hbar}\right)^2 [G_1, G_2] = \frac{i}{\hbar} G_{[1,2]}$$

$$\frac{1}{i\hbar} [G_1, G_2] = G_{[1,2]}$$

which for rotations leads to

$$\frac{1}{i\hbar} \left[\vec{J}.\, \delta_1\vec{\omega},\ \vec{J}.\, \delta_2\vec{\omega}\right] = \vec{J}.\, \delta_{[1,2]}\vec{\omega}$$

16 Observe that $\delta_{[1,2]}\vec{\omega}$ has to be bilinear in $\delta_1\vec{\omega}$ and $\delta_2\vec{\omega}$. In general

$$\left(\delta_{[1,2]}\vec{\omega}\right)_i = g_{imn} (\delta_1\vec{\omega})_m (\delta_2\vec{\omega})_n$$

where g_{imn} once determined, using 15 specify the commutation relations between the angular momentum commutators completely

(17) To determine the group structure constants given for angular momentum we turn back to the specific realization in 3-D space.

$$
\begin{tikzcd}
& \vec{x}_1 \arrow[r, "\delta_2 \vec{\omega}"] & \vec{x}_{21} \\
\vec{x}_0 \arrow[ur, "\delta_1 \vec{\omega}"] \arrow[dr, "\delta_2 \vec{\omega}"'] & & \\
& \vec{x}_2 \arrow[r, "\delta_1 \vec{\omega}"'] & \vec{x}_{12} \arrow[uu, "\delta_{[1,2]} \vec{\omega}"']
\end{tikzcd}
$$

(18)
$$
\begin{aligned}
\vec{x}_{21} &= \vec{x}_1 - \delta_2 \vec{\omega} \times \vec{x}_1 \\
&= \vec{x}_0 - \delta_1 \vec{\omega} \times \vec{x}_0 - \delta_2 \vec{\omega} \times (\vec{x}_0 - \delta_1 \vec{\omega} \times x_0) \\
&= \vec{x}_0 - \delta_1 \vec{\omega} \times \vec{x}_0 - \delta_2 \vec{\omega} \times \vec{x}_0 + \delta_2 \vec{\omega} \times (\delta_1 \vec{\omega} \times \vec{x}_0) \\
&= \vec{x}_0 - \delta_1 \vec{\omega} \times \vec{x}_0 - \delta_2 \vec{\omega} \times \vec{x}_0 + \delta_1 \vec{\omega} \times (\delta_2 \vec{\omega} \times \vec{x}_0)
\end{aligned}
$$

$$
\vec{x}_{12} = \vec{x}_0 - \delta_2 \vec{\omega} \times \vec{x}_0 - \delta_1 \vec{\omega} \times \vec{x}_0 + \delta_1 \vec{\omega} \times (\delta_2 \vec{\omega} \times \vec{x}_0)
$$

(19)
$$
\begin{aligned}
\vec{x}_{21} &= \vec{x}_{12} - \delta_1 \vec{\omega} \times (\delta_2 \vec{\omega} \times \vec{x}_0) + \delta_2 \vec{\omega} \times (\delta_1 \vec{\omega} \times \vec{x}_0) \\
&= \vec{x}_{12} - (\delta_1 \vec{\omega} \times \delta_2 \vec{\omega}) \times \vec{x}_0 \\
&= \vec{x}_{12} - (\delta_1 \vec{\omega} \times \delta_2 \vec{\omega}) \times \vec{x}_{12} \qquad \text{(to leading order)}
\end{aligned}
$$

$$
\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0
$$

$$
\delta_1 \times (\delta_2 \times x) + \delta_2 \times (x \times \delta_1) + x \times (\delta_1 \times \delta_2) = 0
$$

$$
x \times (\delta_1 \times \delta_2) = - \delta_1 \times (\delta_2 \times x) + \delta_2 \times (\delta_1 \times x)
$$

(20) Comparing (19) with

$$
\vec{x}_{21} = \vec{x}_{12} - \delta_{[1,2]} \vec{\omega} \times \vec{x}_{12}
$$

we identify

$$
\delta_{[1,2]} \vec{\omega} = \delta_1 \vec{\omega} \times \delta_2 \vec{\omega} .
$$

$$
\left(\delta_{[1,2]} \vec{\omega} \right)_i = \epsilon_{imn} (\delta_1 \vec{\omega})_m (\delta_2 \vec{\omega})_n .
$$

(21) Since the structure constants should be independent of specific realization, we conclude, using (16) and (20)

$$g_{imn} = \epsilon_{imn}.$$

(22) This when used in (15) completely specifies the commutation relations for the components of the angular momentum as

$$\frac{1}{i\hbar}\left[\vec{J}\cdot\delta_1\vec{\omega},\ \vec{J}\cdot\delta_2\vec{\omega}\right] = \vec{J}\cdot(\delta_1\vec{\omega}\times\delta_2\vec{\omega}),$$

which leads to

$$\frac{1}{i\hbar}\left[J_i,\ J_j\right] = \epsilon_{ijk} J_k$$

(23) Note that the components, which are operators, satisfy

$$\frac{1}{i\hbar}(J_2 J_3 - J_3 J_2) = J_1,$$

$$-\frac{1}{i\hbar}(J_1 J_3 - J_3 J_1) = J_2,$$

$$\frac{1}{i\hbar}(J_1 J_2 - J_2 J_1) = J_3,$$

which can be written as

$$\frac{1}{i\hbar}\vec{J}\times\vec{J} = \vec{J},$$

and is possible only for operator $\vec{J}$'s.

→ Introduce spin-$\frac{1}{2}$.

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$

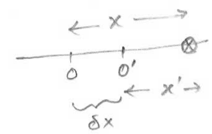
$[\vec{\sigma}\cdot\vec{a}, \vec{\sigma}\cdot\vec{b}] = 2i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

$\frac{1}{i\hbar}\left[\frac{\hbar}{2}\vec{\sigma}\cdot\vec{a}, \frac{\hbar}{2}\vec{\sigma}\cdot\vec{b}\right] = \frac{\hbar}{2}\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

$\Rightarrow \vec{J} = \frac{\hbar}{2}\vec{\sigma}.$

(24) We have studied how state vectors $|\rangle$ transform under rotations. Let us now see how an operator X transforms under an unitary transformation.

Should be introduced before (4) → **Notations:**



$x' = x - \delta x$

$$|\rangle' = |\rangle - \delta(|\rangle) = U^{-1}|\rangle = \left(1 - \frac{i}{\hbar} G\right)|\rangle$$

$$\langle|' = \langle| - \delta(\langle|) = \langle| U = \langle|\left(1 + \frac{i}{\hbar} G\right)$$

(25) Consider $\quad X |\rangle_a = |\rangle_b$.

Under rotations we have.

$$X' |\rangle_a' = |\rangle_b'$$

$$X' U^{-1} |\rangle_a = U^{-1} |\rangle_b$$

$$U X' U^{-1} |\rangle_a = |\rangle_b$$

Thus, we have.

$$X' = U^{-1} X U$$

$$X - \delta X = \left(1 - \frac{i}{\hbar} G\right) X \left(1 - \frac{i}{\hbar} G\right)$$

$$= X - \frac{i}{\hbar}[G, X]$$

$$\Rightarrow \qquad \delta X = \frac{1}{i\hbar}[X, G]$$

26 In 22 we have seen

$$\frac{1}{i\hbar}\left[\vec{J}\cdot\delta_1\vec{\omega}\ ,\ \vec{J}\cdot\delta_2\vec{\omega}\right] = \vec{J}\cdot(\delta_1\vec{\omega}\times\delta_2\vec{\omega})$$

which can be stated in the form.

$$\vec{J}\cdot(\delta_1\vec{\omega}\times\delta_2\vec{\omega}) = \delta_1\vec{\omega}\cdot(\delta_2\vec{\omega}\times\vec{J})$$

$$\frac{1}{i\hbar}\left[\vec{J}\ ,\ \underbrace{\vec{J}\cdot\delta\vec{\omega}}_{G}\right] = \delta\vec{\omega}\times\vec{J}$$

27 Comparing 25 and 26 we learn that

$$\frac{1}{i\hbar}\left[\vec{J},\ \underbrace{\vec{J}\cdot\delta\vec{\omega}}_{G}\right] = \delta\vec{\omega}\times\vec{J} = \delta\vec{J}$$

28 We define a vector operator as a set of three elements that transforms under rotations like the components of $\vec{J}$. Thus, a vector $\vec{V}$ transforms as

$$\frac{1}{i\hbar}\left[\vec{V},\ \vec{J}\cdot\delta\vec{\omega}\right] = \delta\vec{\omega}\times\vec{V}$$

29 Since a scalar S is unchanged under rotations we define

$$\frac{1}{i\hbar}\left[S,\ \vec{J}\cdot\delta\vec{\omega}\right] = 0$$