Homework No. 01 (Spring 2014)

PHYS 420: Electricity and Magnetism II

Due date: Monday, 2014 Jan 27, 4.30pm

1. Using the series representation for Bessel functions,

$$J_m(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!} \left(\frac{t}{2}\right)^{m+2n}, \tag{1}$$

prove the relation

$$J_m(t) = (-1)^m J_{-m}(t). \tag{2}$$

Hint: Break the sum on n into two parts. Note that the gamma function $\Gamma(z)$, which generalizes the factorial,

$$n! = \Gamma(n+1), \qquad \Gamma(z+1) = z\Gamma(z), \tag{3}$$

beyond positive integers, satisfies

$$\frac{1}{\Gamma(-k)} = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots. \tag{4}$$

2. Use the integral representation of $J_m(t)$,

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}, \tag{5}$$

to prove the recurrence relations

$$2\frac{d}{dt}J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{6a}$$

$$2\frac{m}{t}J_m(t) = J_{m-1}(t) + J_{m+1}(t). \tag{6b}$$

3. Using the recurrence relations of Eq. (6), show that

$$\left(-\frac{d}{dt} + \frac{m-1}{t}\right)\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = \left(\frac{d}{dt} + \frac{m+1}{t}\right)\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_m(t) \tag{7}$$

and from this derive the differential equation satisfied by $J_m(t)$.

4. The modified Bessel functions, $I_m(t)$ and $K_m(t)$, satisfy the differential equation

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]\left\{\begin{array}{c} I_m(t) \\ K_m(t) \end{array}\right\} = 0. \tag{8}$$

Derive the identity, for the Wronskian, (upto a constant C)

$$I_m(t)K'_m(t) - K_m(t)I'_m(t) = -\frac{C}{t}, \tag{9}$$

where

$$I'_m(t) \equiv \frac{d}{dt}I_m(t) \qquad \text{and} \qquad K'_m(t) \equiv \frac{d}{dt}K_m(t). \tag{10}$$

Further, determine the value of the constant C on the right hand side of Eq. (9) using the asymptotic forms for the modified Bessel functions:

$$I_m(t) \xrightarrow{t\gg 1} \frac{1}{\sqrt{2\pi}}\frac{e^t}{\sqrt{t}}, \tag{11}$$

$$K_m(t) \xrightarrow{t\gg 1} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}. \tag{12}$$

5. Show that the integral representation for modified Bessel functions,

$$K_m(t) = \int_0^\infty d\theta\, \cosh m\theta\, e^{-t\cosh\theta}, \tag{13}$$

satisfies the differential equation for modified Bessel functions,

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]K_m(t) = 0. \tag{14}$$

6. Verify by substitution that

$$g_m(\rho, \rho'; k) = I_m(k\rho_<)K_m(k\rho_>) \tag{15}$$

satisfies the differential equation

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k^2\right]g_m(\rho, \rho'; k) = \frac{\delta(\rho - \rho')}{\rho}. \tag{16}$$

Hint: Rewrite $g_m(\rho, \rho'; k)$ in terms of sign function $\theta(x)$ and use the identity $d\theta(x)/dx = \delta(x)$.