Homework No. 04 (Spring 2014)

PHYS 420: Electricity and Magnetism II

Due date: Friday, 2014 Mar 7, 4.30pm

1. (Based on Problem 5.58, Griffiths 4th edition.) A circular loop of wire carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about its axis, say z-axis.

   (a) Show that the current density generated by this motion is given by

   $$\mathbf{J}(\mathbf{r}) = \frac{q}{2\pi a}\,\boldsymbol{\omega} \times \mathbf{r}\,\delta(\rho - a)\delta(z - 0). \tag{1}$$

   Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for circular motion.

   (b) Using

   $$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r} \times \mathbf{J}(\mathbf{r}). \tag{2}$$

   determine the magnetic dipole moment of this loop to be

   $$\mathbf{m} = \frac{qa^2}{2}\boldsymbol{\omega}. \tag{3}$$

   (c) Calculate the angular momentum of the rotating loop to be

   $$\mathbf{L} = ma^2\boldsymbol{\omega}, \tag{4}$$

   where m is the mass of the loop.

   (d) What is the gyromagnetic ratio g of the rotating loop, which is defined by the relation $\mathbf{m} = g\mathbf{L}$.

2. A charged spherical shell carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about a diameter, say z-axis.

   (a) Show that the current density generated by this motion is given by

   $$\mathbf{J}(\mathbf{r}) = \frac{q}{4\pi a^2}\,\boldsymbol{\omega} \times \mathbf{r}\,\delta(r - a). \tag{5}$$

   Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$ and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for circular motion.

   (b) Using

   $$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r} \times \mathbf{J}(\mathbf{r}). \tag{6}$$

   determine the magnetic dipole moment of the rotating sphere to be

   $$\mathbf{m} = \frac{qa^2}{3}\boldsymbol{\omega}. \tag{7}$$

3. A steady current I flows down a long cylindrical wire of radius a. The current density in the wire is described by, $n > 0$,

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}}\, \frac{I}{2\pi a^2}\,(n+2)\left(\frac{\rho}{a}\right)^n \theta(a-\rho). \tag{8}$$

   (a) Show that, indeed,

$$\int_S d\mathbf{S}\cdot\mathbf{J}(\mathbf{r}) = I. \tag{9}$$

   (b) Using Ampere's law show that the magnetic field inside and outside the cylinder is given by

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{2I}{\rho}\left(\dfrac{\rho}{a}\right)^{n+2}\hat{\boldsymbol{\phi}} & \rho < R, \\[2ex] \dfrac{\mu_0}{4\pi}\dfrac{2I}{\rho}\,\hat{\boldsymbol{\phi}} & \rho > R. \end{cases} \tag{10}$$

   (c) Plot the magnetic field as a function of ρ.