Homework No. 06 (Spring 2014)

PHYS 420: Electricity and Magnetism II

Due date: Wednesday, 2014 Apr 16, 4.30pm

1. Consider the four-vector $x^\alpha = (ct, \mathbf{x})$. In terms of the proper time, that remains invariant under a Lorentz transformation,

$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x}, \tag{1}$$

the energy E and momentum $\mathbf{p}$ of a particle of mass m is defined as

$$mc^2 \frac{dx^\alpha}{ds} = (E, \mathbf{p}c). \tag{2}$$

Find the explicit expressions for E and $\mathbf{p}$ in terms of $\mathbf{v} = d\mathbf{x}/dt$, c, and m. Show that

$$\frac{dx^\alpha}{ds}\frac{dx_\alpha}{ds} = -1, \tag{3}$$

and use this to derive $E^2 = p^2c^2 + m^2c^4$.

2. In terms of the four-vector potential

$$A^\mu = (\frac{1}{c}\phi, \mathbf{A}) \tag{4}$$

the Maxwell field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{5}$$

and the corresponding dual tensor is defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{6}$$

Derive the following relations, which involve quantities that remain invariant under Lorentz transformations.

(a) $c^2 F^{\mu\nu} F_{\mu\nu} = 2(c^2B^2 - E^2)$.
(b) $c^2 \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} = 2(E^2 - c^2B^2)$.
(c) $cF^{\mu\nu} \tilde{F}_{\mu\nu} = -4\,\mathbf{B} \cdot \mathbf{E}$.