Final Exam (Spring 2014)

PHYS 520B: Electromagnetic Theory

Date: 2014 May 6

1. (**20 points.**) The spherical harmonics

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\left(\frac{e^{i\phi}}{\sin\theta}\right)^m \left(\frac{d}{d\cos\theta}\right)^{l-m} \frac{(\cos^2\theta-1)^l}{2^l l!} \tag{1}$$

satisfy the orthonormality condition

$$\int d\Omega\, Y^*_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{2}$$

Using the relation between the Legendre's polynomial $P_l(x)$ and the spherical harmonics,

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}} Y_{l0}(\theta,\phi), \tag{3}$$

derive the orthonormality condition satisfied by Legendre's polynomials.

2. (**20 points.**) Evalauate the integral

$$\int_{-\infty}^{\infty} dx\, e^{-x^2}\delta(\sin x) \tag{4}$$

as a sum.
Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{5}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$.

3. (**20 points.**) Neglecting quadrupole and higher moments, the angular distribution of power radiated by a non-relativistic particle is given by

$$\frac{dP}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{1}{4\pi c^3}\left[(\hat{\mathbf{r}}\times\ddot{\mathbf{d}})^2 + (\hat{\mathbf{r}}\times\ddot{\boldsymbol{\mu}})^2 + 2\hat{\mathbf{r}}\cdot(\ddot{\mathbf{d}}\times\ddot{\boldsymbol{\mu}})\right]. \tag{6}$$

Calculate the contribution to the total power radiated $P(t)$ from the third term, that represents interference between $\mathbf{d}$ and $\boldsymbol{\mu}$, by integrating over all solid angles.

4. (**20 points.**) An electron of charge e and mass m moves in a circular orbit under the Coulomb forces produced by a proton. Suppose, as it radiates, the electron continues to move on a circle.

   (a) Determine the acceleration a of the electron using Newton's laws of motion.

   (b) Show that the energy of the system is given by

$$E = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{7}$$

   (c) Using the Larmor formula

$$P = -\frac{dE}{dt} = \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3c^3}a^2, \tag{8}$$

   construct a differential equation for E.

   (d) Show that

$$\frac{dE}{dt} = \frac{1}{2}ma\frac{dr}{dt}. \tag{9}$$

   Thus, construct a differential equation for r.

   (e) In a finite time the electron reaches the center. Calculate how long it takes for the electron to hit the proton if it starts from an initial radius of $r_{\text{initial}} = 10^{-10}\,\text{m}$.

   (This instability was one of the reasons for the discovery of quantum mechanics.)

5. (**20 points.**) Consider the motion of three non-relativisitic particles (speed v_i small compared to speed of light c, $v_i \ll c$,) of identical charges $q_i = q$ and identical masses $m_i = m$, $i = 1, 2, 3$. The radiated power by the individual particles are given by the expressions

$$P_i(t) = \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{3c^3}\mathbf{a}_i^2(t_e), \tag{10}$$

   where $\mathbf{a}_i(t_e)$ is the acceleration of the i-th particle at the time of emission

$$t_e = t - \frac{r}{c}. \tag{11}$$

   Let the total contribution to radiated power from the three particles together be denoted by the subscript $(1+2+3)$. Consider the motion of three particles moving on a circle with same uniform speed while remaining at the vertices of a equilateral triangle at each moment. Find the total radiated power $P_{(1+2+3)}(t)$. (Hint: The centripetal acceleration is in the radial direction.)