Homework No. 01 (Spring 2014)

PHYS 520B: Electromagnetic Theory

Due date: Friday, 2014 Jan 24, 4.30pm

1. Show that
$$\boldsymbol{\nabla}\cdot\varepsilon(\rho)\boldsymbol{\nabla} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\varepsilon(\rho)\rho\frac{\partial}{\partial\rho} + \frac{\varepsilon(\rho)}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \varepsilon(\rho)\frac{\partial^2}{\partial z^2}, \tag{1}$$
where (ρ,ϕ,z) are the cylindrical coordinates.

2. Verify by substitution that
$$g_m(t,t') = I_m(t_<)K_m(t_>) \tag{2}$$
satisfies the differential equation
$$\left[-\frac{1}{t}\frac{\partial}{\partial t}t\frac{\partial}{\partial t} + \frac{m^2}{t^2} + 1\right]g_m(t,t') = \frac{\delta(t-t')}{t}. \tag{3}$$

3. The cylindrical Green's function satisfies
$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\varepsilon(\rho)\rho\frac{\partial}{\partial\rho} + \varepsilon(\rho)\frac{m^2}{\rho^2} + \varepsilon(\rho)k_z^2\right]g_m(\rho,\rho';k_z) = \frac{\delta(\rho-\rho')}{\rho}. \tag{4}$$
Consider a dielectric cylinder described by
$$\varepsilon(\rho) = \begin{cases}\varepsilon_2 & \text{for} \quad \rho < a,\\ \varepsilon_1 & \text{for} \quad a < \rho.\end{cases} \tag{5}$$

   (a) Integrate Eq. (4) around $\rho = \rho'$ to derive the continuity conditions:
$$g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{6a}$$
$$\varepsilon(\rho)\rho\frac{\partial}{\partial\rho}g_m(\rho,\rho';k_z)\Bigg|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1. \tag{6b}$$

   (b) Integrate Eq. (4) around $\rho = a$ to derive the continuity conditions:
$$g_m(\rho,\rho';k_z)\Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0, \tag{7a}$$
$$\varepsilon(\rho)\rho\frac{\partial}{\partial\rho}g_m(\rho,\rho';k_z)\Bigg|_{\rho=a-\delta}^{\rho=a+\delta} = 0. \tag{7b}$$

(c) For

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2, & \rho < a, \\ \varepsilon_1, & a < \rho, \end{cases} \tag{8}$$

derive the solution

$\underline{\rho' < a}$:

$$g_m(\rho, \rho'; k_z) = \begin{cases} \frac{1}{\varepsilon_2} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\varepsilon_2} I_m(k_z\rho) I_m(k_z\rho') \frac{K_a K_a'}{\Delta}, & \rho, \rho' < a, \\ \frac{1}{\varepsilon_1} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\varepsilon_1} K_m(k_z\rho) I_m(k_z\rho') \frac{I_a' K_a}{\Delta}, & \rho' < a < \rho. \end{cases} \tag{9}$$

$\underline{a < \rho'}$:

$$g_m(\rho, \rho'; k_z) = \begin{cases} \frac{1}{\varepsilon_2} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\varepsilon_2} I_m(k_z\rho) K_m(k_z\rho') \frac{I_a K_a'}{\Delta}, & \rho < a < \rho', \\ \frac{1}{\varepsilon_1} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\varepsilon_1} K_m(k_z\rho) K_m(k_z\rho') \frac{I_a I_a'}{\Delta}, & a < \rho, \rho'. \end{cases} \tag{10}$$

We used the definitions

$$\frac{1}{\Delta} = \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 I_a K_a' - \varepsilon_2 K_a I_a')}, \qquad I_a \equiv I_m(k_z a), \quad K_a \equiv K_m(k_z a). \tag{11}$$

(d) Show that in the perfect conductor limit

Inside the cylinder

$$g_m(\rho, \rho'; k_z) = \frac{1}{\varepsilon_2} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\varepsilon_2} I_m(k_z\rho) I_m(k_z\rho') \frac{K_a}{I_a}. \tag{12}$$

Outside the cylinder

$$g_m(\rho, \rho'; k_z) = \frac{1}{\varepsilon_1} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\varepsilon_1} K_m(k_z\rho) K_m(k_z\rho') \frac{I_a}{K_a}. \tag{13}$$