Homework No. 02 (Spring 2014)

PHYS 520B: Electromagnetic Theory

Due date: Friday, 2014 Feb 7, 4.30pm

1. The generating function for the spherical harmonics, $Y_{lm}(\theta, \phi)$, is

$$\frac{1}{l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\,Y_{lm}(\theta,\phi)\,\psi_{lm}, \tag{1}$$

where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{2}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \tag{3}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{4}$$

and

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{5}$$

Show that

$$\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right) \tag{6}$$

is unchanged by the substitution: $y_+ \leftrightarrow y_-$, $\theta \to -\theta$, $\phi \to -\phi$. Thus, show that

$$Y_{lm}(\theta,\phi) = Y_{l,-m}(-\theta,-\phi). \tag{7}$$

2. Legendre polynomials of order l is given by (for $|t| < 1$)

$$P_l(t) = \left(\frac{d}{dt}\right)^l \frac{(t^2-1)^l}{2^l l!}. \tag{8}$$

   (a) Write down the explicit forms of the Legendre polynomials $P_l(t)$ for $l = 0, 1, 2, 3$, by completing the l differentiations in Eq. (8).

   (b) Show that the spherical harmonics for $m = 0$ involves the Legendre polynomials,

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta). \tag{9}$$

(c) Using the orthonormality condition for the spherical harmonics

$$\int d\Omega \, Y^*_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{10}$$

recognize the orthogonality statement for Legendre polynomials,

$$\frac{1}{2}\int_{-1}^{1} dt \, P_l(t) P_{l'}(t) = \frac{\delta_{ll'}}{2l+1}. \tag{11}$$

Use

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \tag{12}$$

to check this explicitly for $l, l' = 0, 1, 2$.

3. An example of a null-vector is

$$\mathbf{a} = (-i\cos\alpha, -i\sin\alpha, 1). \tag{13}$$

(a) Identify the corresponding $y_\pm$ in Eq. (3) to show that, now, ψ_{lm} in Eq. (1) is

$$\psi_{lm} = \frac{e^{-im\left(\alpha - \frac{\pi}{2}\right)}}{\sqrt{(l+m)!(l-m)!}}. \tag{14}$$

(b) Then, integrate Eq. (1) to derive an integral representation for spherical harmonics,

$$\frac{1}{l!}\int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{im\alpha} \big[\cos\theta - i\sin\theta\cos(\phi - \alpha)\big]^l = \sqrt{\frac{4\pi}{2l+1}} \frac{i^m \, Y_{lm}(\theta,\phi)}{\sqrt{(l+m)!(l-m)!}}. \tag{15}$$

(c) By setting $m = 0$ derive the corresponding integral representation for Legendre polynomial $P_l(\cos\theta)$:

$$\int_0^{\pi} \frac{d\alpha}{\pi} \big[\cos\theta - i\sin\theta\cos\alpha\big]^l = P_l(\cos\theta). \tag{16}$$

(d) Use the integral representation for $J_0(t)$,

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{it\cos\alpha}, \tag{17}$$

to show that

$$P_l(\cos\theta) = \left(\cos\theta - \sin\theta \frac{d}{dt}\right)^l J_0(t)\Big|_{t=0}. \tag{18}$$

Verify this for $l = 0, 1, 2$.

(e) Now let $\theta = x/l$ and, for fixed x, consider the limit $l \to \infty$, to obtain

$$\lim_{l\to\infty} P_l\left(\cos\frac{x}{l}\right) = J_0(x), \tag{19}$$

which is often used in the approximate form

$$\theta \ll 1, l \gg 1: \qquad P_l(\cos\theta) \sim J_0(l\theta). \tag{20}$$

(f) For what geometrical reason does one expect an asymptotic connection between spherical and cylindrical coordinate functions?