Homework No. 03 (Spring 2014)

PHYS 520B: Electromagnetic Theory

Due date: Thursday, 2014 Feb 20, 4.30pm

1. The solution to the Maxwell equations for the case of magnetostatics was found to be

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{1}$$

Verify that the above solution satisfies magnetostatics equations, that is it satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{2}$$

and

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \, \mathbf{J}. \tag{3}$$

2. (Based on Problem 5.8, Griffiths 4th edition.) The magnetic field at position $\mathbf{r} = (x, y, z)$ due to a finite wire segment of length $2L$ carrying a steady current I, with the caveat that it is unrealistic (why?), placed on the z-axis with its end points at $(0, 0, L)$ and $(0, 0, -L)$, is

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \, \frac{\mu_0 I}{4\pi} \frac{1}{\sqrt{x^2 + y^2}} \left[\frac{z + L}{\sqrt{x^2 + y^2 + (z + L)^2}} - \frac{z - L}{\sqrt{x^2 + y^2 + (z - L)^2}} \right], \tag{4}$$

where $\hat{\boldsymbol{\phi}} = (-\sin\phi \, \hat{\mathbf{i}} + \cos\phi \, \hat{\mathbf{j}}) = (-y \, \hat{\mathbf{i}} + x \, \hat{\mathbf{j}})/\sqrt{x^2 + y^2}$.

(a) Show that by taking the limit $L \to \infty$ we obtain the magnetic field near a long straight wire carrying a steady current I,

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \, \frac{\mu_0 I}{2\pi\rho}, \tag{5}$$

where $\rho = \sqrt{x^2 + y^2}$ is the perpendicular distance from the wire.

(b) Show that the magnetic field on a line bisecting the wire segment is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \, \frac{\mu_0 I}{2\pi\rho} \frac{L}{\sqrt{\rho^2 + L^2}}. \tag{6}$$

(c) Find the magnetic field at the center of a square loop, which carries a steady current I. Let $2L$ be the length of a side, ρ be the distance from center to side, and $R = \sqrt{\rho^2 + L^2}$ be the distance from center to a corner. (Caution: Notation differs from Griffiths.) You should obtain

$$B = \frac{\mu_0 I}{2R} \frac{4}{\pi} \tan\frac{\pi}{4}. \tag{7}$$

(d) Show that the magnetic field at the center of a regular n-sided polygon, carrying a steady current I is

$$B = \frac{\mu_0 I}{2R}\frac{n}{\pi}\tan\frac{\pi}{n}, \tag{8}$$

where R is the distance from center to a corner of the polygon.

(e) Show that the magnetic field at the center of a circular loop of radius R,

$$B = \frac{\mu_0 I}{2R}, \tag{9}$$

is obtained in the limit $n \to \infty$.

3. In 1935 Fritz London and Heinz London proposed that a superconductor is characterized by the equations

$$\mu_0 \frac{\partial \mathbf{j}_s}{\partial t} = \frac{1}{\lambda_L^2}\mathbf{E}, \tag{10}$$

$$\mu_0 \boldsymbol{\nabla} \times \mathbf{j}_s = -\frac{1}{\lambda_L^2}\mathbf{B}, \tag{11}$$

where $\mathbf{j}_s$ is the superconducting current density.

(a) Show that the above equations can be expressed in terms of the single equation

$$\mu_0 \mathbf{j}_s = -\frac{1}{\lambda_L^2}\mathbf{A} + \boldsymbol{\nabla}\chi. \tag{12}$$

(Hint: Choose the scalar potential $\phi = \lambda_L^2 \partial\chi/\partial t$.)

(b) Assuming $\partial\mathbf{B}/\partial t = 0$, show that

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}, \tag{13}$$

which implies the Meissner effect, that a uniform magnetic field cannot exist inside a superconductor. λ_L is called the London penetration depth, and has a typical value of 10^{-5} cm.