Homework No. 04 (Spring 2014)

PHYS 520B: Electromagnetic Theory

Due date: Friday, 2014 Mar 7, 4.30pm

1. Consider a wire segment of length $2L$ carrying a steady current I, described by

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}} I \delta(x)\delta(y)\theta(-L < z < L), \tag{1}$$

when the rod is placed on the z-axis centered on the origin. Here $\theta(-L < z < L) = 0$, if $z > L$ and $z < -L$, and $\theta(-L < z < L) = 1$, otherwise.

(a) Show that the vector potential of the wire is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \hat{\mathbf{z}} I \left[\sinh^{-1}\left(\frac{L-z}{\sqrt{x^2+y^2}} \right) + \sinh^{-1}\left(\frac{L+z}{\sqrt{x^2+y^2}} \right) \right]. \tag{2}$$

(b) Show that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}). \tag{3}$$

(c) Thus, express the vector potential of Eq. (2) in the form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \hat{\mathbf{z}} I \left[-2\ln\frac{R}{L} + F\left(\frac{z}{L}, \frac{R}{L} \right) \right], \tag{4}$$

where $R^2 = x^2 + y^2$ and

$$F(a,b) = \ln[1 - a + \sqrt{(1-a)^2 + b^2}] + \ln[1 + a + \sqrt{(1+a)^2 + b^2}]. \tag{5}$$

(d) Show that

$$\mathbf{A}(\mathbf{r}) \xrightarrow{R \ll L, z \ll L} -\frac{\mu_0}{4\pi} \hat{\mathbf{z}}\, 2I \ln\frac{R}{2L}. \tag{6}$$

(e) Using $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ determine the magnetic field for an infinite rod (placed on the z-axis) to be

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{2I}{R} \hat{\boldsymbol{\phi}}. \tag{7}$$

2. A charged spherical shell carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about a diameter, say z-axis.

(a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{4\pi a^2} \boldsymbol{\omega} \times \mathbf{r}\, \delta(r-a). \tag{8}$$

Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$ and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for circular motion.

(b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{J}(\mathbf{r}). \tag{9}$$

determine the magnetic dipole moment of the rotating sphere to be

$$\mathbf{m} = \frac{qa^2}{3}\boldsymbol{\omega}. \tag{10}$$

(c) Calculate the vector potential $\mathbf{A}(\mathbf{r})$ inside and outside the sphere, without choosing $\mathbf{r}$ to be along $\hat{\mathbf{z}}$.
Hint: Use ideas in lecture notes of Set 23. In particular, observe that

$$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \\ &= \frac{1}{2}\sqrt{\frac{8\pi}{3}}\Big[-Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\Big]\hat{\mathbf{i}} + \frac{1}{2i}\sqrt{\frac{8\pi}{3}}\Big[-Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\Big]\hat{\mathbf{j}} \\ &\quad + \sqrt{\frac{4\pi}{3}}Y_{10}(\theta,\phi)\hat{\mathbf{k}}.\end{aligned} \tag{11}$$

3. A circular loop of wire carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about its axis, say z-axis.

(a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{2\pi a}\,\boldsymbol{\omega}\times\mathbf{r}\,\delta(\rho - a)\delta(z-0). \tag{12}$$

Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ for circular motion.

(b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{J}(\mathbf{r}). \tag{13}$$

determine the magnetic dipole moment of this loop to be

$$\mathbf{m} = \frac{qa^2}{2}\boldsymbol{\omega}. \tag{14}$$

(c) Calculate the vector potential $\mathbf{A}(0,0,z)$ on the z-axis.

(d) Calculate the magnetic field $\mathbf{B}(0,0,z)$ on the z-axis.