Homework No. 05 (Spring 2014)

PHYS 520B: Electromagnetic Theory

Due date: Friday, 2014 Mar 28, 4.30pm

1. From Maxwell's equations, without introducing potentials, show that the electric and magnetic fields satisfy the inhomogeneous wave equations

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\boldsymbol{\nabla}\rho(\mathbf{r},t) - \frac{1}{\varepsilon_0}\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t), \tag{1a}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}(\mathbf{r},t) = \mu_0\boldsymbol{\nabla}\times\mathbf{J}(\mathbf{r},t). \tag{1b}$$

   Extend this result to magnetic charges and currents.

2. The 4-dimensional Euclidean Green's function satisfies

$$-\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_4^2}\right) G_E(x_1,\ldots,x_4) = \delta(x_1)\cdots\delta(x_4). \tag{2}$$

   (a) Show that the solution to this equation can be written as the Fourier transform

$$G_E(x_1,\ldots,x_4) = \int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{dk_4}{2\pi}\,\frac{e^{i(k_1x_1+\ldots+k_4x_4)}}{k_1^2+\ldots+k_4^2}. \tag{3}$$

   (b) Verify the integral

$$\frac{1}{M} = \int_0^\infty ds\, e^{-sM}. \tag{4}$$

   (c) Using Eq. (4) in Eq. (3) show that

$$G_E(x_1,\ldots,x_4) = \int_0^\infty ds \prod_{m=1}^{4}\left[\int_{-\infty}^{\infty}\frac{dk_m}{2\pi}e^{-sk_m^2+ik_mx_m}\right]. \tag{5}$$

   (d) Show that

$$\int_{-\infty}^{\infty} dk_m e^{-sk_m^2+ik_mx_m} = \sqrt{\frac{\pi}{s}}\,e^{-\frac{x_m^2}{4s}} \tag{6}$$

   (e) Using the integral of Eq. (6) in Eq. (5) and using the integral representation of Gamma function,

$$\Gamma(z) = \int_0^\infty \frac{ds}{s}s^z e^{-s}, \tag{7}$$

   show that

$$G_E(x_1,\ldots,x_4) = \frac{1}{4\pi^2}\frac{1}{x_1^2+\ldots+x_4^2}. \tag{8}$$

(f) By making the complex replacement

$$x_4 \to ict \equiv \lim_{\varepsilon \to 0+} e^{i(\frac{\pi}{2}-\varepsilon)} ct, \tag{9}$$

show that

$$D_+(\mathbf{r}, t) = iG_E(\mathbf{r}, ict) \tag{10}$$

satisfies the differential equation

$$\left(-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) D_+(\mathbf{r}, t) = \delta^{(3)}(\mathbf{r}) \delta(ct), \tag{11}$$

where $\mathbf{r} = (x_1, x_2, x_3)$, with the corresponding solution

$$D_+(\mathbf{r}, t) = \lim_{\varepsilon \to 0+} \frac{i}{4\pi^2} \frac{1}{\mathbf{r}^2 - (ct)^2 + i\varepsilon'}, \tag{12}$$

where $\varepsilon' = (ct)^2 \varepsilon$

(g) Using the δ-function representation

$$\pi\delta(x) = \lim_{\varepsilon \to 0+} \frac{\varepsilon}{x^2 + \varepsilon^2} = \lim_{\varepsilon \to 0+} \mathrm{Im} \frac{1}{x - i\varepsilon} \tag{13}$$

and the identity

$$\delta(r^2 - (ct)^2) = \frac{1}{2r} \big[\delta(r - ct) + \delta(r + ct) \big] \tag{14}$$

show that

$$\mathrm{Re}\, D_+(\mathbf{r}, t) = \frac{1}{2} \left[\frac{\delta(r - ct)}{4\pi r} + \frac{\delta(r + ct)}{4\pi r} \right], \tag{15}$$

where the two terms here are the retarded and advanced Green's functions, respectively, up to numerical factors.

(h) Refer problem 31.9 in Schwinger et al. for further discussion on this subject. (Will not be graded.)

3. Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left| \frac{dF}{dx} \Big|_{x=a_r} \right|}, \tag{16}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, evaluate

$$\delta(\sin x), \quad \delta(\cos x), \quad \text{and} \quad \delta(\tan x). \tag{17}$$