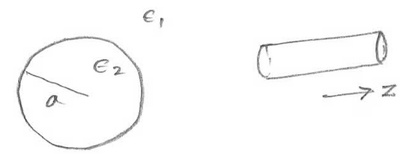
Green's function for a dielectric cylinder

(1) Consider a (right circular) cylindrically symmetric dielectric medium described by

$$\epsilon(\rho) = \begin{cases} \epsilon_2 & \rho < a \\ \epsilon_1 & a < \rho \end{cases}$$



(2) The corresponding Green's function satisfies the differential equation

$$-\vec{\nabla} \cdot \epsilon(\rho)\, \vec{\nabla}\, G(\vec{r}, \vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

(3) The cylindrical symmetry suggests the form

$$G(\vec{r}, \vec{r}\,') = \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')}\, g_m(\rho, \rho'; k_z),$$

which when substituted in (2) leads to the reduced differential equation for the Green's function $g_m(\rho, \rho'; k_z)$,

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\, \epsilon(\rho)\, \rho \frac{\partial}{\partial\rho} + \epsilon(\rho)\frac{m^2}{\rho^2} + \epsilon(\rho)\, k_z^2 \right] g_m(\rho, \rho'; k_z) = \frac{\delta(\rho-\rho')}{\rho}.$$

(4) We used

$$\vec{\nabla} = \hat{\rho}\,\frac{\partial}{\partial\rho} + \hat{\phi}\,\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\,\frac{\partial}{\partial z}$$

HW

to evaluate

$$-\vec{\nabla}\cdot\epsilon(\rho)\,\vec{\nabla} = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\,\epsilon(\rho)\,\rho\,\frac{\partial}{\partial\rho} + \frac{\epsilon(\rho)}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \epsilon(\rho)\,\frac{\partial^2}{\partial z^2}.$$

(5) For the dielectric medium of (1) the reduced Green's function is given using

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k_z^2\right] g_m(\rho,\rho';k_z) = \begin{cases} \dfrac{1}{\epsilon_2}\,\dfrac{\delta(\rho-\rho')}{\rho} & \rho < a \\[2mm] \dfrac{1}{\epsilon_1}\,\dfrac{\delta(\rho-\rho')}{\rho} & a < \rho \end{cases}$$

(6) The continuity conditions satisfied by the reduced Green's function are.

HW

at $\rho = \rho'$:

(i) $$\epsilon(\rho)\,\rho\,\frac{\partial}{\partial\rho}\,g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1$$

(ii) $$g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0$$

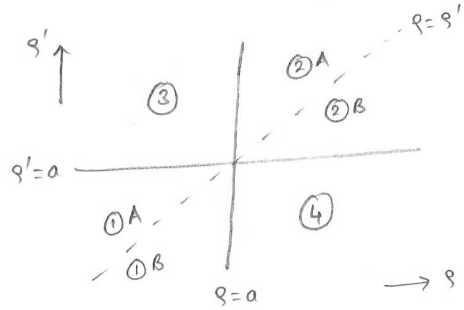
at $\rho = a$

(i) $$\epsilon(\rho)\,\rho\,\frac{\partial}{\partial\rho}\,g_m(\rho,\rho';k_z)\Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0$$

(ii) $$g_m(\rho,\rho';k_z)\Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0$$

7 We will consider the case $\underline{\underline{a < \rho'}}$. The second case $\rho' < a$ will be a homework problem.

HW $\longrightarrow$



8 For $\underline{\underline{a < \rho'}}$ we have the solutions to be of the form

$$g_m(\rho, \rho'; k_z) = \begin{cases} \overset{=0}{\cancel{A}}\, K_m(k_z\rho) + B\, I_m(k_z\rho) & \rho < a < \rho' \quad (3) \\ C\, K_m(k_z\rho) + D\, I_m(k_z\rho) & a < \rho < \rho' \quad (2A) \\ E\, K_m(k_z\rho) + \overset{=0}{\cancel{F}}\, I_m(k_z\rho) & a < \rho' < \rho \quad (2B) \end{cases}$$

9 Using 8 in the continuity conditions of 6 at $\rho = a$ we have.

$$C\, K_m(k_z a) + D\, I_m(k_z a) = B\, I_m(k_z a)$$

$$\epsilon_1 C\, K_m'(k_z a) + \epsilon_1 D\, I_m'(k_z a) = \epsilon_2 B\, I_m'(k_z a)$$

10 Rewriting 9 we have.

$$B\, I_m(k_z a) - C\, K_m(k_z a) = D\, I_m(k_z a)$$

$$\epsilon_2 B\, I_m'(k_z a) - \epsilon_1 C\, K_m'(k_z a) = \epsilon_1 D\, I_m'(k_z a)$$

which leads to

$$B = D \frac{\left[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_1 K_m(k_z a) I_m'(k_z a)\right]}{\left[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)\right]}$$

$$C = D \frac{(\epsilon_2 - \epsilon_1)\, I_m(k_z a) I_m'(k_z a)}{\left[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)\right]}$$

11 We have restrained from using the wronskian.

$$I_m(t) K_m'(t) - K_m(t) I_m'(t) = -\frac{1}{t}.$$

12 Using 8 in the continuity condition of 6 at $\rho = \rho'$ we have.

$$E\, K_m(k_z \rho') - C\, K_m(k_z \rho') - D\, I_m(k_z \rho') = 0 \qquad \text{— (i)}$$

$$E\, K_m'(k_z \rho') - C\, K_m'(k_z \rho') - D\, I_m'(k_z \rho') = -\frac{1}{\epsilon_1}\frac{1}{k_z \rho'} \qquad \text{— (ii)}$$

13 Multiplying 12 - (i) by $K_m'(k_z \rho')$ and 12 - (ii) by $K_m(k_z \rho')$ and subtracting the two we have

$$-D\, I_m(k_z \rho')\, K_m'(k_z \rho') + D\, K_m(k_z \rho')\, I_m'(k_z \rho') = \frac{1}{\epsilon_1}\frac{1}{k_z \rho'} K_m(k_z \rho'),$$

which using the Wronskian of 11 leads to

$$D = \frac{1}{\epsilon_1} K_m(k_z \rho')$$

14 Using D of 13 and C of 10 in 12 - (i) we find.

$$E = \frac{1}{\epsilon_1} I_m(k_z \rho') + C$$

15 Using 14 in 8 we have for $\underline{a < \varrho' < \varrho}$ $\rightarrow$ 2 B

$$g_m(\varrho,\varrho';k_z) = \frac{1}{\epsilon_1} I_m(k_z\varrho') K_m(k_z\varrho) - \frac{1}{\epsilon_1} K_m(k_z\varrho') K_m(k_z\varrho) \frac{(\epsilon_1-\epsilon_2) I_m(k_z a) I_m'(k_z a)}{[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)]}$$

16 Using C of 10 and D of 13 in 8 for $\underline{a < \varrho < \varrho'}$ $\rightarrow$ 2 A

$$g_m(\varrho,\varrho';k_z) = \frac{1}{\epsilon_1} I_m(k_z\varrho) K_m(k_z\varrho') - \frac{1}{\epsilon_1} K_m(k_z\varrho') K_m(k_z\varrho) \frac{(\epsilon_1-\epsilon_2) I_m(k_z a) I_m'(k_z a)}{[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)]}$$

17 Together, 15 and 16 has the form for $\underline{a < \varrho, \varrho'}$ $\rightarrow$ 2

$$g_m(\varrho,\varrho';k_z) = \frac{1}{\epsilon_1} I_m(k_z\varrho_<) K_m(k_z\varrho_>) - \frac{1}{\epsilon_1} K_m(k_z\varrho) K_m(k_z\varrho') \frac{(\epsilon_1-\epsilon_2) I_m(k_z a) I_m'(k_z a)}{[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)]}$$

18 Using B of 10 in 8 we have for $\underline{\varrho < a < \varrho'}$ $\rightarrow$ 3

$$g_m(\varrho,\varrho';k_z) = \frac{1}{\epsilon_1} I_m(k_z\varrho) K_m(k_z\varrho') \frac{[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_1 K_m(k_z a) I_m'(k_z a)]}{[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)]}$$

19 We rewrite 18 in the form. $\underline{\rho < a < \rho'}$

$$g_m(\rho,\rho';k_z) = \frac{1}{\epsilon_2} I_m(k_z\rho)\, K_m(k_z\rho')$$

$$- \frac{1}{\epsilon_1} I_m(k_z\rho)\, K_m(k_z\rho') \left[\frac{\epsilon_1}{\epsilon_2} - \frac{\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_1 K_m(k_z a) I_m'(k_z a)}{\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)} \right]$$

$$= \frac{1}{\epsilon_2} I_m(k_z\rho) K_m(k_z\rho') - \frac{1}{\epsilon_2} I_m(k_z\rho) K_m(k_z\rho') \frac{(\epsilon_1 - \epsilon_2)\, I_m(k_z a)\, K_m'(k_z a)}{\left[\epsilon_1 I_m(k_z a)\, K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)\right]}$$

20 Using 17 and 19 we have for $\underline{\underline{a < \rho'}}$

$$g_m(\rho,\rho';k_z) = \begin{cases} \frac{1}{\epsilon_2} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\epsilon_2} I_m(k_z\rho) K_m(k_z\rho') \frac{(\epsilon_1-\epsilon_2) I_m(k_z a) K_m'(k_z a)}{\left[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)\right]}, & \rho < a < \rho' \;\to 3 \\ \frac{1}{\epsilon_1} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\epsilon_1} K_m(k_z\rho) K_m(k_z\rho') \frac{(\epsilon_1-\epsilon_2) I_m(k_z a) I_m'(k_z a)}{\left[\epsilon_1 I_m(k_z a) K_m'(k_z a) - \epsilon_2 K_m(k_z a) I_m'(k_z a)\right]}, & a < \rho, \rho' \;\to 2\ A \,\&\, B \end{cases}$$

$\leftarrow$ HW

(21) In HW we find the solution for $\rho' < a$.

In total we have.

Notation: $I_a = I_m(k_z a)$
$K_a = K_m(k_z a)$

$\underline{\rho' < a}$

$$g_m(\rho,\rho';k_z) = \begin{cases} \frac{1}{\epsilon_2} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\epsilon_2} I_m(k_z\rho) I_m(k_z\rho') \frac{(\epsilon_1-\epsilon_2) K_a K_a'}{(\epsilon_1 I_a K_a' - \epsilon_2 K_a I_a')} & \rho,\rho' < a \quad (1) \\ \frac{1}{\epsilon_1} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\epsilon_1} K_m(k_z\rho) I_m(k_z\rho') \frac{(\epsilon_1-\epsilon_2) I_a' K_a}{(\epsilon_1 I_a K_a' - \epsilon_2 K_a I_a')} & \rho' < a < \rho \quad (4) \end{cases}$$

$\underline{a < \rho'}$

$$g_m(\rho,\rho';k_z) = \begin{cases} \frac{1}{\epsilon_2} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\epsilon_2} I_m(k_z\rho) K_m(k_z\rho') \frac{(\epsilon_1-\epsilon_2) I_a K_a'}{(\epsilon_1 I_a K_a' - \epsilon_2 K_a I_a')} & \rho < a < \rho' \quad (3) \\ \frac{1}{\epsilon_1} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\epsilon_1} K_m(k_z\rho) K_m(k_z\rho') \frac{(\epsilon_1-\epsilon_2) I_a I_a'}{(\epsilon_1 I_a K_a' - \epsilon_2 K_a I_a')} & a < \rho,\rho' \quad (2) \end{cases}$$

(22) In the perfect conductor limit

Inside the cylinder: $\rho,\rho' < a$ (obtained by taking $\epsilon_1 \to \infty$ in region (1))

$$g_m(\rho,\rho';k_z) = \frac{1}{\epsilon_2} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\epsilon_2} I_m(k_z\rho) I_m(k_z\rho') \frac{K_a}{I_a}.$$

Outside the cylinder: $a < \rho,\rho'$ (obtained by taking $\epsilon_2 \to \infty$ in region (2))

$$g_m(\rho,\rho';k_z) = \frac{1}{\epsilon_1} I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{1}{\epsilon_1} K_m(k_z\rho) K_m(k_z\rho') \frac{I_a}{K_a}$$