Set - 21

Interaction energy between a charge and dielectric/conductor

(1) For two charges we have. q_1 q_2.

$$E = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

(2) For three charges

$$E_{int} = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|} + \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|} \right]$$

$$= \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{3} \sum_{j=1}^{3} \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

$i \neq j$: removes the self interaction terms

$\frac{1}{2}$: removes the double counting

(3) For a continuous charge distribution the above generalizes to

$$E_{int} = \frac{1}{2} \int d^3r \int d^3r' \frac{\rho(\vec{r})\, \rho(\vec{r}')}{|\vec{r} - \vec{r}'|} - E_{self}.$$

$$= \frac{1}{2} \int d^3r \int d^3r' \, \rho(\vec{r})\, G_0(\vec{r}, \vec{r}')\, \rho(\vec{r}') - E_{self}.$$

where E_{self} is the self energy, which does not contribute to physical quantities. Note that $G_0(\vec{r}, \vec{r}')$ is the free Green's function.

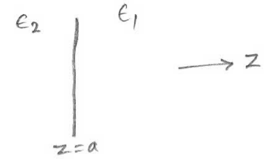
(4) In the presence of a dielectric the Green's function is modified.

$$G_0(\vec{r},\vec{r}') \longrightarrow G(\vec{r},\vec{r}')$$

$G_0(\vec{r},\vec{r}')$: in the absence of dielectric

$G(\vec{r},\vec{r}')$: in the presence of dielectric.

For example, for a planar dielectric



we have.

$$G(\vec{r},\vec{r}') = \frac{1}{\epsilon_1}\int \frac{d^2k}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\Big[\underbrace{\frac{1}{2k_\perp}e^{-k_\perp|z-z'|}}_{\text{free Green's function.}} - \underbrace{\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\,\frac{1}{2k_\perp}\,e^{-k_\perp|z-a|}\,e^{-k_\perp|z'-a|}}_{\text{additional contribution due to presence of dielectric.}}\Big]$$

(5) Note that, in the presence of a dielectric, a charge can interact with itself, or more precisely it can interact with the dielectric, which is equivalent to interact with an "image charge". Thus, the charge in interacting energy due to the presence of dielectric will be

$$E = \frac{1}{2}\int d^3r\int d^3r'\, \rho(\vec{r})\,\big[\,G(\vec{r},\vec{r}') - G_0(\vec{r},\vec{r}')\big]\,\rho(\vec{r}')$$

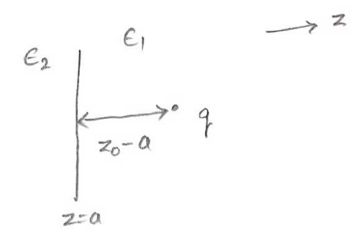
(6) In particular, for a point charge described by

$$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r}-\vec{r}_0)$$

we have.

$$E = \frac{1}{2} q^2 \left[G(\vec{r},\vec{r}\,') - G_0(\vec{r},\vec{r}\,') \right]_{\vec{r}=\vec{r}\,'=\vec{r}_0}$$

(7) For the case.



$$G(\vec{r},\vec{r}\,') - G_0(\vec{r},\vec{r}\,') = -\frac{1}{\epsilon_1}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp}\, \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\, \frac{1}{2k_\perp}\, e^{-k_\perp|z-a|}\, e^{-k_\perp|z'-a|}$$

$$\left[G(\vec{r},\vec{r}\,') - G_0(\vec{r},\vec{r}\,')\right]_{\vec{r}=\vec{r}\,'=\vec{r}_0} = -\frac{1}{\epsilon_1}\,\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_2}\int \frac{d^2k_\perp}{(2\pi)^2}\,\frac{1}{2k_\perp}\, e^{-2k_\perp|z_0-a|}$$

$$= -\frac{1}{\epsilon_1}\,\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\,\frac{1}{2\pi}\int_0^\infty \cancel{k_\perp}\, dk_\perp\, \frac{1}{2\cancel{k_\perp}}\, e^{-2k_\perp|z_0-a|}$$

$$= -\frac{1}{4\pi\epsilon_1}\,\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\,\frac{1}{2|z_0-a|}$$

(8) Using (7) in (6)

$$E = -\frac{1}{4\pi\epsilon_1}\,\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\,\frac{q^2}{4|z_0-a|}$$

9 The force acting on the charge is given by

$$F = -\frac{\partial}{\partial z_0} E$$

$$= \frac{1}{4\pi\epsilon_1} \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} \frac{\partial}{\partial z_0} \frac{q^2}{4|z_0-a|}$$

$$= -\frac{1}{4\pi\epsilon_1} \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} \frac{q^2}{4|z_0-a|^2} \quad \text{for } z_0 > a.$$

10 Comparing the above expression with the force between two charges q and q_{image}

$$F = \frac{1}{4\pi\epsilon_1} \frac{q\, q_{image}}{r^2}$$

r — distance between q and q_{image}.

we can interpret

$$q_{image} = -q \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}$$

$$r = 2|z_0-a|$$

11 Perfect conductor limit : $(\epsilon_2 \to \infty)$

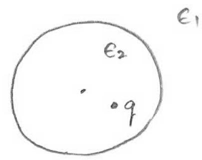
$$E = -\frac{1}{4\pi\epsilon_1} \frac{q^2}{4|z_0-a|}$$

$$F = -\frac{1}{4\pi\epsilon_1} \frac{q^2}{4|z_0-a|^2}.$$

$$q_{image} = -q$$

$$r = 2|z_0-a|.$$

(12) Let us now consider the case of a charge inside a dielectric cylinder.



$$\rho(\vec{r}) = q\, \delta^{(3)}(\vec{r} - \vec{r}_0)$$

$$\vec{r}_0 = (\rho_0, \phi_0, z_0)$$

$$E = \frac{1}{2} q^2 \left[G(\vec{r}, \vec{r}\,') - G_0(\vec{r}, \vec{r}\,') \right]_{\vec{r} = \vec{r}\,' = \vec{r}_0}$$

$$= \frac{1}{2} q^2 \frac{1}{\epsilon_2} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi} e^{im(\phi-\phi')}$$

$$\times (-1) \; I_m(k_z\rho)\, I_m(k_z\rho') \frac{(\epsilon_1 - \epsilon_2)\, K_a K_a'}{\epsilon_1 I_a K_a' - \epsilon_2 K_a I_a'} \Bigg|_{\vec{r} = \vec{r}\,' = \vec{r}_0}$$

$$= -\frac{q^2}{2\epsilon_2} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi} \left[I_m(k_z \rho_0) \right]^2 \frac{(\epsilon_1 - \epsilon_2)\, K_m(k_z a)\, K_m'(k_z a)}{\epsilon_1 I_m(k_z a)\, K_m'(k_z a) - \epsilon_2 K_m(k_z a)\, I_m'(k_z a)}$$

(13) Let the charge be on the axis, then

$\rho_0 = 0$,

$$E = -\frac{q^2}{4\pi\epsilon_2} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \sum_{m=-\infty}^{+\infty} \left[I_m(0) \right]^2 \frac{(\epsilon_1 - \epsilon_2)\, K_m(k_z a)\, K_m'(k_z a)}{\epsilon_1 I_m(k_z a)\, K_m'(k_z a) - \epsilon_2 K_m(k_z a)\, I_m'(k_z a)}$$

(14) Observing that

$$I_m(0) = \begin{cases} 1 & \text{for} \quad m = 0 \\ 0 & \text{for} \quad m > 0 \end{cases}$$

we conclude

$$I_m(0) = \delta_{m0}$$

which leads to

$$E = -\frac{q^2}{4\pi\epsilon_2} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \frac{(\epsilon_1 - \epsilon_2)\; K_0(k_z a)\, K_0'(k_z a)}{\epsilon_1\, I_0(k_z a)\, K_0'(k_z a) - \epsilon_2\, K_0(k_z a)\, I_m'(k_z a)}$$

$$= -\frac{q^2}{4\pi\epsilon_2} \frac{1}{2\pi a} \int_{-\infty}^{+\infty} dt \frac{(\epsilon_1 - \epsilon_2)\; K_0(t)\, K_0'(t)}{\epsilon_1\, I_0(t)\, K_0'(t) - \epsilon_2\, K_0(t)\, I_0'(t)}$$

$$= -\frac{q^2}{4\pi\epsilon_2} \frac{1}{\pi a} \int_{0}^{\infty} dt \frac{(\epsilon_1 - \epsilon_2)\; K_0(t)\, K_0'(t)}{\epsilon_1\, I_0(t)\, K_0'(t) - \epsilon_2\, K_0(t)\, I_0'(t)}$$

(15) Let us next specialize to the case of a perfect conductor outside the cylinder,

$$\epsilon_1 \to \infty,$$

which leads to, using (14)

$$E = -\frac{q^2}{4\pi\epsilon_2} \frac{1}{\pi a} \int_0^{\infty} dt \frac{K_0(t)}{I_0(t)}$$

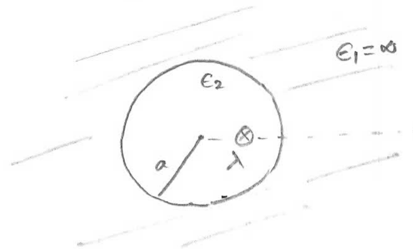
16 We can numerically evaluate, for example using Mathematica,

$$\int_0^\infty dt \, \frac{K_0(t)}{I_0(t)} = 1.36768$$

17 Using 16 in 15 we have.

$$E = - \frac{q^2}{4\pi \epsilon_2} \frac{1.36768}{\pi a}.$$

(18) Let us next consider a line charge inside a perfectly conducting cylinder



$$\rho(\vec{r}) = \lambda \frac{\delta(\rho - \rho_0)}{\rho_0} \delta(\phi - 0)$$

(19)
$$E = \frac{1}{2} \int d^3r \int d^3r' \, \rho(\vec{r}) \left[G(\vec{r}, \vec{r}') - G_0(\vec{r}, \vec{r}') \right] \rho(\vec{r}')$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dz' \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \int_0^{\infty} \rho \, d\rho \int_0^{\infty} \rho' \, d\rho' \; \lambda \frac{\delta(\rho - \rho_0)}{\rho_0} \; \lambda \frac{\delta(\rho' - \rho_0)}{\rho_0} \delta(\phi) \delta(\phi')$$

$$\times \frac{(-1)}{\epsilon_2} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi} e^{im(\phi-\phi')} I_m(k_z\rho) I_m(k_z\rho') \frac{K_a}{I_a}$$

$$= -\frac{\lambda^2}{2\epsilon_2} \underbrace{\int_{-\infty}^{+\infty} dz'}_{L} \int_{-\infty}^{+\infty} \frac{dk_z}{\cancel{2\pi}} \cancel{2\pi} \, \delta(k_z) \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi} \left[I_m(k_z \rho_0) \right]^2 \frac{K_m(k_z a)}{I_m(k_z a)}$$

$$\frac{E}{L} = -\frac{\lambda^2}{4\pi\epsilon_2} \sum_{m=-\infty}^{+\infty} \left[I_m(0) \right]^2 \; \underset{t \to 0}{Lt} \; \frac{K_m\left(t \frac{a}{\rho_0}\right)}{I_m\left(t \frac{a}{\rho_0}\right)} \qquad I_m(0) = \delta_{m0}$$

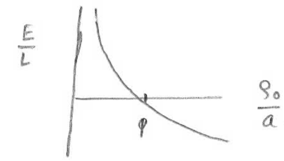
$$= -\frac{\lambda^2}{4\pi\epsilon_2} \; \underset{t \to 0}{Lt} \; K_0\left(t \frac{a}{\rho_0}\right)$$

$$= -\frac{\lambda^2}{4\pi\epsilon_2} \; \underset{t \to 0}{Lt} \; \ln \frac{2\rho_0}{ta}.$$

$$= -\frac{\lambda^2}{4\pi\epsilon_0} \ln \frac{\rho_0}{a} - \frac{\lambda^2}{4\pi\epsilon_2} \; \underset{t \to 0}{Lt} \; \ln \frac{2}{t}$$

unphysical.

(20) $$\frac{E}{L} = -\frac{\lambda^2}{4\pi\epsilon_0} \ln \frac{\rho_0}{a}.$$



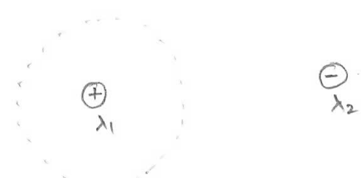
$\rho_0 < a$ inside a cylinder.

Since energy is minimized, the wire would like t move away from the center towards the surface of cylinder. The force is

$$\frac{F}{L} = -\frac{\partial}{\partial \rho_0} \frac{E}{L}$$

$$= \frac{\lambda^2}{4\pi\epsilon_0} \frac{1}{\rho_0}$$

(21) Let us consider two wires.



$$\rho_i(\vec{r}) = \lambda_i \frac{\delta(\rho-\rho_i)}{\rho_i} \delta(\phi_i)$$

Using Gauss's law

E-electric field.

$$E\,(2\pi\rho)\,L = \frac{1}{\epsilon_0} \int_{-\infty}^{+\infty} dz \int_0^{2\pi} d\phi \int_0^{\infty} \rho\, d\rho\; \lambda_1 \frac{\delta(\rho-\rho_1)}{\rho_1} \delta(\phi)$$

$$= \frac{1}{\epsilon_0} L \lambda_1$$

$$\vec{E} = \frac{\lambda_1}{2\pi\epsilon_0} \frac{1}{\rho} \hat{\rho}$$

(22)

Force $\vec{F} = \int d^3r \; \rho_2(\vec{r}) \; \vec{E}(\vec{r})$

$$= -L \int_0^{2\pi} d\phi \int_0^{2\pi} \rho \, d\rho \;\; \lambda_2 \, \frac{\delta(\rho-\rho_2)}{\rho_2} \; \delta(\phi) \; \frac{\lambda_1}{2\pi\epsilon_0} \; \frac{1}{\rho} \; \hat{\rho}.$$

force on λ_2 ← $\frac{\vec{F}}{L} = -\frac{\lambda_1 \lambda_2}{2\pi\epsilon_0} \; \frac{1}{\rho_2} \; \hat{\rho}.$ → force on λ_1 will be positive.

(23) Thus we can interpret

$$\frac{F}{L} = \frac{\lambda^2}{4\pi\epsilon_0} \; \frac{1}{\rho_0} = \frac{\lambda \; \lambda_{image}}{2\pi\epsilon_0} \; \frac{1}{\rho}$$

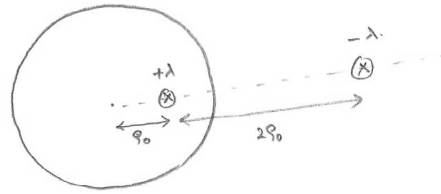
ρ → distance between λ & λ_{image}.

$$\Rightarrow \quad \frac{\lambda_{image}}{\rho} = \frac{\lambda}{2\rho_0}$$

Thus, we can interpret

$$\lambda_{image} = -\lambda$$

$$\rho = 2\rho_0$$



→ Note that image could be inside the cylinder.

→ We can verify that the tangential component of electric field is zero on the conduct. Refer problem 2.8 in Jackson.