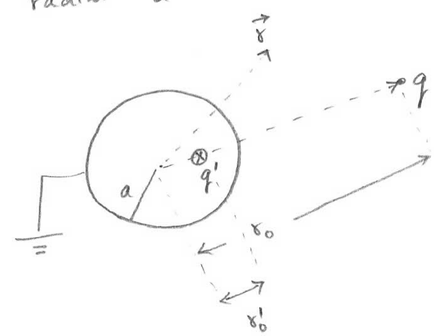
SET-24

Method of images - Conducting sphere

(1) Consider a charge q outside a perfectly conducting sphere of radius a



(2) Electric field is zero inside a conductor, and can have only components normal to the surface. Any constant electric potential will achieve this. A grounded conductor means that this constant is chosen to be zero. Thus, we have.

$$\phi(\vec{a}) = 0 \quad \text{(grounded conductor.)}$$

(3) Using symmetry arguments we conclude:

(i) Image charge is on the radial line.

(ii) Image charge is opposite in sign.

(4) $$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|} - \frac{1}{4\pi\epsilon_0}\frac{q'}{|\vec{r}-\vec{r}_0'|}$$

(5) $\phi(\vec{a}) = 0$

$\vec{r} \cdot \vec{r}_0 = 2 r r_0 \cos\gamma$

$$\Rightarrow \quad \frac{q}{|\vec{a} - \vec{r}_0|} - \frac{q'}{|\vec{a} - \vec{r}_0'|} = 0$$

$$\frac{q}{\sqrt{a^2 + r_0^2 - 2 a r_0 \cos\gamma}} = \frac{q'}{\sqrt{a^2 + r_0'^2 - 2 a r_0' \cos\gamma}}$$

$$q^2 (a^2 + r_0'^2 - 2 a r_0' \cos\gamma) = q'^2 (a^2 + r_0^2 - 2 a r_0 \cos\gamma)$$

$$q^2 (a^2 + r_0'^2) - q'^2 (a^2 + r_0^2) + 2a (r_0 q'^2 - r_0' q^2) \cos\gamma = 0$$

(6) Since the above equation is true for all γ, we have.

$$q^2 (a^2 + r_0'^2) = q'^2 (a^2 + r_0^2) \quad \text{— (i)}$$

and

$$r_0 q'^2 - r_0' q^2 = 0 \quad \text{— (ii)}$$

(7) Thus, we have.

$$\frac{q'^2}{q^2} = \frac{r_0'}{r_0}$$

(8) Using (7) in (6) - (i)

$$r_0 (a^2 + r_0'^2) = r_0' (a^2 + r_0^2)$$

$$\left(\frac{r_0'}{r_0}\right)^2 - \frac{r_0'}{r_0}\left(1 + \frac{a^2}{r_0^2}\right) + \frac{a^2}{r_0^2} = 0$$

$$\frac{r_0'}{r_0} = \frac{1}{2}\left(1 + \frac{a^2}{r_0^2}\right) \pm \frac{1}{2}\sqrt{\left(1 + \frac{a^2}{r_0^2}\right)^2 - 4\frac{a^2}{r_0^2}}$$

$$= \frac{1}{2}\left(1 + \frac{a^2}{r_0^2}\right) \pm \frac{1}{2}\left(1 - \frac{a^2}{r_0^2}\right) = \begin{cases} 1 & \longrightarrow \text{trivial.} \\ \dfrac{a^2}{r_0^2} \end{cases}$$

(9) Using (8) in (7) we have.

$$\frac{q'^2}{q^2} = \frac{r_0'}{r_0} = \frac{a^2}{r_0^2}$$

(10) Using (9) in (4)

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\gamma}} - \frac{1}{4\pi\epsilon_0}\frac{q'}{\sqrt{r^2 + r_0'^2 - 2rr_0'\cos\gamma}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\gamma}} - \frac{1}{4\pi\epsilon_0}\frac{\frac{a}{r_0}q}{\sqrt{r^2 + \frac{a^4}{r_0^4}r_0^2 - 2rr_0\frac{a^2}{r_0^2}\cos\gamma}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\gamma}} - \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2\frac{r_0^2}{a^2} + a^2 - 2rr_0\cos\gamma}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\gamma}} - \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{\left(\frac{r}{a}r_0\right)^2 + a^2 - 2rr_0\cos\gamma}}$$

which is clearly zero at $r = a$.

(11) Charge density on the surface of a perfect conductor:

$$\vec{\nabla}\cdot(\epsilon_0 \vec{E}) = \rho - \vec{\nabla}\cdot\vec{P}$$

(12) Let

$$\vec{P} = \vec{P}_0 \,\theta(R-r)$$

$$\vec{\nabla}\cdot\vec{P} = (\vec{\nabla}\cdot\vec{P}_0)\,\theta(R-r) + \vec{P}_0\cdot\vec{\nabla}\theta(R-r)$$

$$= \underbrace{(\vec{\nabla}\cdot\vec{P}_0)\,\theta(R-r)}_{\text{this term is necessarily zero for a perfect conductor}} + \underbrace{(\vec{P}_0\cdot\hat{r})}_{\sigma(\theta,\phi)}\,\delta(r-R)$$

(13) In general, we will have.

$$\rho_{eff}(\vec{r}) = -\vec{\nabla}\cdot\vec{P} = +\sigma(\theta,\phi)\,\delta(\xi-\xi_0)$$

→ ξ is the coordinate of the surface of conductor.

→ For a conductor, we can only consider one side of surface, unless it is a infinitesimally thin conductor.

→ Examples: Surface charge density on a conducting plane $\} = +\sigma(x,y)\,\delta(z-z_0)$

Surface charge density on a conducting sphere $\} = +\sigma(\theta,\phi)\,\delta(r-r_0)$

(14) For a conducting sphere we have.

$\perp \rightarrow$ perpendicular to normal.

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = + \sigma(\theta,\phi)\, \delta(r-a)$$

$$\epsilon_0 \frac{\partial}{\partial r} E_r + \epsilon_0 \underbrace{\vec{\nabla}_{\perp} \cdot \vec{E}_{\perp}}_{=0 \text{ on the conductor}} = + \sigma(\theta,\phi)\, \delta(r-a)$$

$$\epsilon_0 \frac{\partial}{\partial r} E_r = + \sigma(\theta,\phi)\, \delta(r-a)$$

Integrating around $r=a$, we have.

$$\epsilon_0 \int_{r=a-\delta}^{r=a+\delta} dr \frac{\partial}{\partial r} E_r = + \sigma(\theta,\phi)$$

$$E_r(a+\delta) - \underbrace{E_r(a-\delta)}_{=0 \text{ inside a conductor}} = + \frac{1}{\epsilon_0} \sigma(\theta,\phi)$$

$$\sigma(\theta,\phi) = + \epsilon_0 E_r(a+\delta)$$

$$= - \epsilon_0 \frac{\partial}{\partial r} \phi \Big|_{r=a+\delta}$$

(15) Using (10) in (14)

$$\sigma(\theta,\phi) = -\frac{1}{4\pi} \frac{\partial}{\partial r} \left[\frac{q}{\sqrt{r^2 + r_0^2 - 2 r r_0 \cos\gamma}} - \frac{q}{\sqrt{r^2 \frac{r_0^2}{a^2} + a^2 - 2 r r_0 \cos\gamma}} \right] \Bigg|_{r=a+\delta}$$

$$= -\frac{q}{4\pi} \left[\frac{(r_0 \cos\gamma - r)}{(r^2 + r_0^2 - 2 r r_0 \cos\gamma)^{3/2}} - \frac{\left(r_0 \cos\gamma - r \frac{r_0^2}{a^2}\right)}{\left(r^2 \frac{r_0^2}{a^2} + a^2 - 2 r r_0 \cos\gamma\right)^{3/2}} \right]_{r=a+\delta}$$

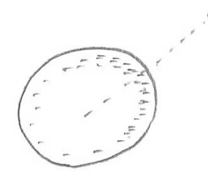
$$= -\frac{q}{4\pi} \frac{a\left(\frac{r_0^2}{a^2} - 1\right)}{(a^2 + r_0^2 - 2 a r_0 \cos\gamma)^{3/2}}$$

(16)
$$\sigma(\theta,\phi) = -\frac{q}{4\pi a^2}\,\frac{\left(\frac{r_0^2}{a^2}-1\right)}{\left(1+\frac{r_0^2}{a^2}-2\frac{r_0}{a}\cos\gamma\right)^{\frac{3}{2}}}$$

$$= -\frac{q}{4\pi a^2}\,\frac{\frac{a}{r_0}\left(1-\frac{a^2}{r_0^2}\right)}{\left(1+\frac{a^2}{r_0^2}-2\frac{a}{r_0}\cos\gamma\right)^{\frac{3}{2}}}$$

(17)
$$\sigma(0,\phi) = -\frac{q}{4\pi a^2}\,\frac{\frac{a}{r_0}\left(1-\frac{a^2}{r_0^2}\right)}{\left(1-\frac{a_0}{r_0}\right)^3} = -\frac{q}{4\pi a^2}\,\frac{\frac{a}{r_0}\left(1+\frac{a}{r_0}\right)}{\left(1-\frac{a}{r_0}\right)^2}$$

$$\sigma(\pi,\phi) = -\frac{q}{4\pi a^2}\,\frac{\frac{a}{r_0}\left(1-\frac{a^2}{r_0^2}\right)}{\left(1+\frac{a}{r_0}\right)^3} = -\frac{q}{4\pi a^2}\,\frac{\frac{a}{r_0}\left(1-\frac{a}{r_0}\right)}{\left(1+\frac{a}{r_0}\right)^2}$$



(18) Note that,

$$\frac{1+\frac{a}{r_0}}{\left(1-\frac{a}{r_0}\right)^2} > 1 \quad \text{and} \quad \frac{1-\frac{a}{r_0}}{\left(1+\frac{a}{r_0}\right)^2} < 1$$

(19) To do: Multipole expansion of $\sigma(\theta,\phi)$.

(20)
$$\sigma_{tot} = a^2\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\ \sigma(\theta,\phi)$$

choose q along z-axis.

$$= 2\pi a^2 \int_0^{\pi} \sin\theta\, d\theta\ \sigma(\theta,\phi)$$

$$= 2\pi a^2(-1)\,\frac{q}{4\pi a^2}\,\frac{\frac{a}{r_0}\left(1-\frac{a^2}{r_0^2}\right)}{\left(1+\frac{a^2}{r_0^2}\right)^{3/2}}\int_0^{\pi}\sin\theta\, d\theta\cdot\frac{1}{\left[1-\frac{2\frac{a}{r_0}}{\left(1+\frac{a^2}{r_0^2}\right)}\cos\theta\right]^{\frac{3}{2}}}$$

(21) $$\sigma_{tot} = -\frac{q}{2}\,\frac{\frac{a}{r_0}\left(1-\frac{a^2}{r_0^2}\right)}{\left(1+\frac{a^2}{r_0^2}\right)^{3/2}}\int_{-1}^{1} dt\,\frac{1}{\left[1-\left(\frac{2\frac{a}{r_0}}{1+\frac{a^2}{r_0^2}}\right)t\right]^{3/2}}.$$

(22) $$\int_{-1}^{1} dt\,\frac{1}{(1-xt)^{3/2}} = -\int_{1+x}^{1-x}\frac{ds}{x}\,\frac{1}{s^{3/2}}.$$

$1-xt=s$

$-x\,dt=ds.$

$$= \frac{1}{x}\int_{1-x}^{1+x}\frac{ds}{s^{3/2}}$$

$$= \frac{1}{x}(-2)\frac{1}{\sqrt{s}}\Bigg|_{s=1-x}^{s=1+x} = -\frac{2}{x}\left[\frac{1}{\sqrt{1+x}}-\frac{1}{\sqrt{1-x}}\right]$$

(23) $$\sigma_{tot} = -\frac{q}{2}\,\frac{\cancel{\frac{a}{r_0}}\left(1-\frac{a^2}{r_0^2}\right)}{\left(1+\frac{a^2}{r_0^2}\right)^{3/2}}(-1)\,\frac{\cancel{2}\left(1+\frac{a^2}{r_0^2}\right)}{\cancel{2}\,\cancel{\frac{a}{r_0}}}\left[\frac{1}{\sqrt{1+\left(\frac{2\frac{a}{r_0}}{1+\frac{a^2}{r_0^2}}\right)}}-\frac{1}{\sqrt{1-\left(\frac{2\frac{a}{r_0}}{1+\frac{a^2}{r_0^2}}\right)}}\right]$$

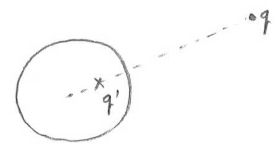
$$= \frac{q}{2}\,\frac{\left(1-\frac{a^2}{r_0^2}\right)}{\left(1+\frac{a^2}{r_0^2}\right)^{1/2}}\left[\frac{\sqrt{1+\frac{a^2}{r_0^2}}}{\left(1+\frac{a}{r_0}\right)}-\frac{\sqrt{1+\frac{a^2}{r_0^2}}}{\left(1-\frac{a}{r_0}\right)}\right]$$

$$= \frac{q}{2}\left(1-\frac{a^2}{r_0^2}\right)\left[\frac{1}{\left(1+\frac{a}{r_0}\right)}-\frac{1}{\left(1-\frac{a}{r_0}\right)}\right]$$

$$= \frac{q}{2}\left(1-\frac{a^2}{r_0^2}\right)\frac{\left(-2\frac{a}{r_0}\right)}{\left(1-\frac{a^2}{r_0^2}\right)}$$

$$= -q\,\frac{a}{r_0}.$$

24



$$q' = -q \frac{a}{r_0}$$

$$r_0' = r_0 \frac{a^2}{r_0^2}.$$

$$F = \frac{1}{4\pi\epsilon_0} \frac{q q'}{|r_0 - r_0'|^2} = -\frac{q^2}{4\pi\epsilon_0} \frac{\frac{a}{r_0}}{\left| r_0 - r_0 \frac{a^2}{r_0^2} \right|^2} = -\frac{q^2}{4\pi\epsilon_0} \frac{a r_0}{(r_0^2 - a^2)^2}$$

$$E_{int} = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \frac{q q'}{|r_0 - r_0'|} = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{2} \frac{\frac{a}{r_0}}{\left| r_0 - r_0 \frac{a^2}{r_0^2} \right|} = -\frac{1}{2} \frac{q^2}{4\pi\epsilon_0} \frac{a}{(r_0^2 - a^2)}$$

$$F = -\frac{\partial}{\partial r_0} E_{int}$$

$$= -\frac{\partial}{\partial r_0} \frac{(-1)}{2} \frac{q^2}{4\pi\epsilon_0} \frac{a}{(r_0^2 - a^2)}$$

$$= \frac{1}{2} \frac{q^2}{4\pi\epsilon_0} (-1) \frac{a}{(r_0^2 - a^2)^2} 2 r_0.$$

$$= -\frac{q^2}{4\pi\epsilon_0} \frac{a r_0}{(r_0^2 - a^2)^2}$$ ✓