Rotating charged spherical shell

① A uniformly charged spherical shell is described by the charge density

$$\rho(\vec{r}) = \frac{Q}{4\pi a^2}\,\delta(r-a)$$

② The current density due to motion will be $\quad \vec{j} = q\vec{v} \rightarrow \rho\vec{v}$

$$\vec{J}(\vec{r}) = \rho(\vec{r})\,\vec{v}$$

$$= \frac{Q}{4\pi a^2}\,\vec{v}\,\delta(r-a)$$

$$= \frac{Q}{4\pi a^2}\,\vec{\omega}\times\vec{r}\,\delta(r-a)$$

③ Here we used $\vec{v} = \vec{\omega}\times\vec{r}$, which is applicable for rigid rotations: $\quad \boxed{\omega = \frac{d\phi}{dt}}$

$$\frac{d\hat{r}}{dt} = \omega\,\hat{\phi}$$

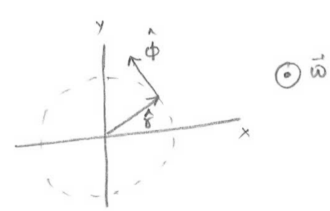
$$\vec{r} = r\,\hat{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{dr}{dt}}_{=0}\,\hat{r} + r\,\frac{d\hat{r}}{dt}$$

for rotation about constant radius.

$$= r\,\omega\,\hat{\phi}$$

$$= \vec{\omega}\times\hat{r}$$



4 $$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

$$= \frac{\mu_0}{4\pi} \frac{Q}{4\pi a^2} \int_0^\infty r'^2 dr' \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \; \frac{\vec{\omega} \times \vec{r}\,' \; \delta(r'-a)}{\sqrt{r^2 + r'^2 - 2\,\vec{r}\cdot\vec{r}\,'}}$$

$$= \frac{\mu_0}{4\pi} \frac{Q}{4\pi} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \; \frac{\vec{\omega} \times \vec{r}\,'}{r^2 + r'^2 - 2\,\vec{r}\cdot\vec{r}\,'} \Bigg|_{\vec{r}\,' = \vec{a}}$$

on sphere.

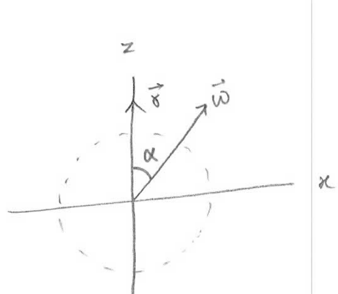
5 We have three vectors: $\vec{\omega}$, $\vec{r}$, $\vec{r}\,' = \vec{a}$

6 Let us choose $\vec{r}$ to be along the z-axis. In homework you will redo this calculation without this restriction.

$\vec{r} = (0, 0, r) \longrightarrow$ chosen along z-axis

$\vec{\omega} = (\omega \sin\alpha, 0, \omega\cos\alpha) \longrightarrow$ chosen in x-z plane

$\vec{a} = \vec{r}\,' = a\,(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$



$\vec{\omega} \times \vec{r}\,' = \vec{\omega} \times \left[a\sin\theta'\cos\phi' \,\hat{i} + a\sin\theta'\sin\phi' \,\hat{j} + a\cos\theta' \,\hat{k} \right]$

$\vec{r}\cdot\vec{r}\,' = ar\cos\theta'$

(7) Using (6) in (4) we have.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{Qa}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \; \frac{\vec{\omega}\times[\sin\theta'\cos\phi'\,\hat{i} + \sin\theta'\sin\phi'\,\hat{j} + \cos\theta'\,\hat{k}]}{\sqrt{r^2 + a^2 - 2ar\cos\theta'}}$$

(8) Evaluating the ϕ' integral we have, using

$$\int_0^{2\pi} d\phi' \cos\phi' = 0 \quad \text{and} \quad \int_0^{2\pi} d\phi' \sin\phi' = 0,$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{Qa}{2}(\vec{\omega}\times\hat{k}) \int_0^{\pi} \sin\theta' d\theta' \frac{\cos\theta'}{\sqrt{r^2 + a^2 - 2ar\cos\theta'}}$$

(9) We have earlier shown that, substitute $t = r^2 + a^2 - 2ar\cos\theta'$,

$$\int_0^{\pi} \sin\theta' d\theta' \frac{\cos\theta'}{\sqrt{r^2 + a^2 - 2ar\cos\theta'}} = \int_{(r-a)^2}^{(r+a)^2} \frac{dt}{2ar}\frac{1}{\sqrt{t}}\frac{(r^2 + a^2 - t^2)}{2ar}$$

$$= \begin{cases} \dfrac{2}{3}\dfrac{r}{a^2} & r < a \\[2ex] \dfrac{2}{3}\dfrac{a}{r^2} & a < r \end{cases}$$

(10) Using (9) in (8) we have.

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{Qa^2}{3}\dfrac{\vec{\omega}\times r\hat{k}}{a^3} & r < a \\[2ex] \dfrac{\mu_0}{4\pi}\dfrac{Qa^2}{3}\dfrac{\vec{\omega}\times r\hat{k}}{r^3} & a < r \end{cases}$$

(11) Releasing the contraint that $\vec{r} = r\hat{k}$, which is verified when done without any constraint to begin with as you do in your homework, we have.

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0}{4\pi} \dfrac{Qa^2}{3} \dfrac{\vec{\omega} \times \vec{r}}{a^3} & r < a, \\[2ex] \dfrac{\mu_0}{4\pi} \dfrac{Qa^2}{3} \dfrac{\vec{\omega} \times \vec{r}}{r^3} & a < r. \end{cases}$$

(12) Comparing this with the vector potential for a point magnetic dipole we identify

$$\vec{m} = \frac{Qa^2}{3} \vec{\omega}.$$

(13) The magnetic field is calculated,

$$\begin{aligned} \vec{B}(\vec{r}) &= \vec{\nabla} \times \vec{A}(\vec{r}) \\ &= \begin{cases} \dfrac{\mu_0}{4\pi} \dfrac{Qa^2}{3} \dfrac{\vec{\nabla} \times (\vec{\omega} \times \vec{r})}{a^3} & r < a \\[2ex] \dfrac{\mu_0}{4\pi} \dfrac{Qa^2}{3} \vec{\nabla} \times \left(\dfrac{\vec{\omega} \times \vec{r}}{r^3} \right) & a < r \end{cases} \end{aligned}$$

(14)
$$\begin{aligned} \vec{\nabla} \times (\vec{\omega} \times \vec{r}) &= \vec{\omega}\,(\vec{\nabla} \cdot \vec{r}) - \vec{\omega} \cdot \vec{\nabla}\, \vec{r} \\ &= 3\vec{\omega} - \vec{\omega} \\ &= 2\vec{\omega} \end{aligned}$$

(15)
$$\vec{\nabla} \times \left(\frac{\vec{\omega} \times \vec{r}}{r^3} \right) = \vec{\omega} \; \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3} \right) - \vec{\omega} \cdot \vec{\nabla} \; \frac{\vec{r}}{r^3}$$

$$= \vec{\omega} \left[\frac{3}{r^3} - \frac{3}{r^3} \right] - \vec{\omega} \cdot \left[\frac{\overleftrightarrow{1}}{r^3} - 3 \, \frac{\vec{r}\,\vec{r}}{r^5} \right]$$

$$= \vec{\omega} \; 4\pi \; \delta^{(3)}(\vec{r}) + \frac{1}{r^3} \left[3 \, (\vec{m} \cdot \hat{r}) \, \hat{r} - \vec{m} \right]$$

this term does not contribute for $0 < a < r$.

(16) Using (14) and (15) in (13) we have.

$$\vec{B}(\vec{r}) = \begin{cases} \dfrac{\mu_0}{4\pi} \; 2 \vec{m} \, , & r < a, \\[2ex] \dfrac{\mu_0}{4\pi} \; \dfrac{1}{r^3} \left[3 (\vec{m} \cdot \hat{r}) \, \hat{r} - \vec{m} \right] , & a < r. \end{cases}$$