Take-Home Exam No. 02 (Spring 2014)

PHYS 530A: Quantum Mechanics II

Due date: Monday, 2014 Apr 14

In Einsteinian relativity the new feature is the finiteness of c, speed of light, which requires the abandonment of absolute simultaneity. For infinitesimal Poincaré transformations, Einsteinian relativity is characterized by

$$t' = t - \frac{1}{c}\delta\varepsilon^0 - \frac{1}{c^2}\delta\mathbf{v}\cdot\mathbf{r}, \tag{1a}$$

$$\mathbf{r}' = \mathbf{r} - \delta\boldsymbol{\varepsilon} - \delta\boldsymbol{\omega}\times\mathbf{r} - \delta\mathbf{v}t, \tag{1b}$$

where $\delta\varepsilon^0/c$ corresponds to translation in time, $\delta\boldsymbol{\varepsilon}$ corresponds to translation in space, $\delta\boldsymbol{\omega}$ corresponds to a rotation, and $\delta\mathbf{v}$ corresponds to boost.

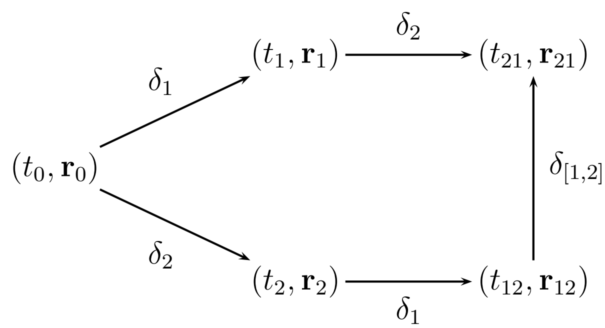

Figure 1: Group transformation

1. Derive the composition properties of the Poincaré group:

$$\delta_{[1,2]}\varepsilon^0 = \frac{1}{c}(\delta_1\mathbf{v}\cdot\delta_2\boldsymbol{\varepsilon} - \delta_2\mathbf{v}\cdot\delta_1\boldsymbol{\varepsilon}), \tag{2a}$$

$$\delta_{[1,2]}\boldsymbol{\varepsilon} = (\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\varepsilon} - \delta_2\boldsymbol{\omega}\times\delta_1\boldsymbol{\varepsilon}) + \frac{1}{c}(\delta_1\mathbf{v}\delta_2\varepsilon^0 - \delta_2\mathbf{v}\delta_1\varepsilon^0), \tag{2b}$$

$$\delta_{[1,2]}\boldsymbol{\omega} = \delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\omega} - \frac{1}{c^2}\delta_1\mathbf{v}\times\delta_2\mathbf{v}, \tag{2c}$$

$$\delta_{[1,2]}\mathbf{v} = (\delta_1\mathbf{v}\times\delta_2\boldsymbol{\omega} - \delta_2\mathbf{v}\times\delta_1\boldsymbol{\omega}). \tag{2d}$$

2. The generators of the unitary transformation induced by the above infinitesimal coordinate transformations are comprised in

$$G = -\frac{1}{c}\delta\varepsilon^0 cP^0 + \delta\boldsymbol{\varepsilon}\cdot\mathbf{P} + \delta\boldsymbol{\omega}\cdot\mathbf{J} + \delta\mathbf{v}\cdot\mathbf{N} + \delta\phi 1. \tag{3}$$

The generator of time translation in Einsteinian relativity is cP^0, which corresponds with the Galilean generator for time translation H, the Hamiltonian, by the relation

$$cP^0 = H + Mc^2. \tag{4}$$

Using the group commutation relations

$$\frac{1}{i\hbar}\big[G_1, G_2\big] = \delta_{[1,2]}G \tag{5}$$

derive

$$\begin{aligned}
\frac{1}{i\hbar}\Big[-\delta_1\varepsilon^0 P^0 &+ \delta_1\boldsymbol{\varepsilon}\cdot\mathbf{P} + \delta_1\boldsymbol{\omega}\cdot\mathbf{J} + \delta_1\mathbf{v}\cdot\mathbf{N},\ -\delta_2\varepsilon^0 P^0 + \delta_2\boldsymbol{\varepsilon}\cdot\mathbf{P} + \delta_2\boldsymbol{\omega}\cdot\mathbf{J} + \delta_2\mathbf{v}\cdot\mathbf{N}\Big] \\
&= -\delta_{[1,2]}\varepsilon^0 P^0 + \delta_{[1,2]}\boldsymbol{\varepsilon}\cdot\mathbf{P} + \delta_{[1,2]}\boldsymbol{\omega}\cdot\mathbf{J} + \delta_{[1,2]}\mathbf{v}\cdot\mathbf{N} && \text{(6a)} \\
&= -\frac{1}{c}(\delta_1\mathbf{v}\cdot\delta_2\boldsymbol{\varepsilon} - \delta_2\mathbf{v}\cdot\delta_1\boldsymbol{\varepsilon})P^0 \\
&\quad +(\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\varepsilon} - \delta_2\boldsymbol{\omega}\times\delta_1\boldsymbol{\varepsilon})\cdot\mathbf{P} + \frac{1}{c}(\delta_1\mathbf{v}\delta_2\varepsilon^0 - \delta_2\mathbf{v}\delta_1\varepsilon^0)\cdot\mathbf{P} \\
&\quad +(\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\omega})\cdot\mathbf{J} - \frac{1}{c^2}(\delta_1\mathbf{v}\times\delta_2\mathbf{v})\cdot\mathbf{J} \\
&\quad +(\delta_1\mathbf{v}\times\delta_2\boldsymbol{\omega} - \delta_2\mathbf{v}\times\delta_1\boldsymbol{\omega})\cdot\mathbf{N}. && \text{(6b)}
\end{aligned}$$

Here we have used

$$\delta_{[1,2]}\phi = 0, \tag{7}$$

see [1].

3. (a) Show that the response of the generators under time translation is given by

$$\begin{aligned}
\frac{1}{i\hbar}\big[-\delta_1\varepsilon^0 P^0, -\delta_2\varepsilon^0 P^0\big] &= 0, && \text{(8a)} \\
\frac{1}{i\hbar}\big[\delta_1\boldsymbol{\varepsilon}\cdot\mathbf{P}, -\delta_2\varepsilon^0 P^0\big] &= 0, && \text{(8b)} \\
\frac{1}{i\hbar}\big[\delta_1\boldsymbol{\omega}\cdot\mathbf{J}, -\delta_2\varepsilon^0 P^0\big] &= 0, && \text{(8c)} \\
\frac{1}{i\hbar}\big[\delta_1\mathbf{v}\cdot\mathbf{N}, -\delta_2\varepsilon^0 P^0\big] &= \frac{1}{c}\delta_2\varepsilon^0\delta_1\mathbf{v}\cdot\mathbf{P}. && \text{(8d)}
\end{aligned}$$

(b) Show that the response of the generators under translation in space is given by

$$\begin{aligned}
\frac{1}{i\hbar}\big[-\delta_1\varepsilon^0 P^0, \delta_2\boldsymbol{\varepsilon}\cdot\mathbf{P}\big] &= 0, && \text{(9a)} \\
\frac{1}{i\hbar}\big[\delta_1\boldsymbol{\varepsilon}\cdot\mathbf{P}, \delta_2\boldsymbol{\varepsilon}\cdot\mathbf{P}\big] &= 0, && \text{(9b)} \\
\frac{1}{i\hbar}\big[\delta_1\boldsymbol{\omega}\cdot\mathbf{J}, \delta_2\boldsymbol{\varepsilon}\cdot\mathbf{P}\big] &= (\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\varepsilon})\cdot\mathbf{P}, && \text{(9c)} \\
\frac{1}{i\hbar}\big[\delta_1\mathbf{v}\cdot\mathbf{N}, \delta_2\boldsymbol{\varepsilon}\cdot\mathbf{P}\big] &= -\frac{1}{c}\delta_1\mathbf{v}\cdot\delta_2\boldsymbol{\varepsilon}P^0. && \text{(9d)}
\end{aligned}$$

(c) Show that the response of the generators under rotation is given by

$$\frac{1}{i\hbar}\big[-\delta_1\varepsilon^0 P^0, \delta_2\boldsymbol{\omega}\cdot\mathbf{J}\big] = 0, \tag{10a}$$

$$\frac{1}{i\hbar}\big[\delta_1\boldsymbol{\varepsilon}\cdot\mathbf{P}, \delta_2\boldsymbol{\omega}\cdot\mathbf{J}\big] = (\delta_1\boldsymbol{\varepsilon}\times\delta_2\boldsymbol{\omega})\cdot\mathbf{P}, \tag{10b}$$

$$\frac{1}{i\hbar}\big[\delta_1\boldsymbol{\omega}\cdot\mathbf{J}, \delta_2\boldsymbol{\omega}\cdot\mathbf{J}\big] = (\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\omega})\cdot\mathbf{J}, \tag{10c}$$

$$\frac{1}{i\hbar}\big[\delta_1\mathbf{v}\cdot\mathbf{N}, \delta_2\boldsymbol{\omega}\cdot\mathbf{J}\big] = (\delta_1\mathbf{v}\times\delta_2\boldsymbol{\omega})\cdot\mathbf{N}. \tag{10d}$$

(d) Show that the response of the generators under boost is given by

$$\frac{1}{i\hbar}\big[-\delta_1\varepsilon^0 P^0, \delta_2\mathbf{v}\cdot\mathbf{N}\big] = -\frac{1}{c}\delta_1\varepsilon^0\delta_2\mathbf{v}\cdot\mathbf{P}, \tag{11a}$$

$$\frac{1}{i\hbar}\big[\delta_1\boldsymbol{\varepsilon}\cdot\mathbf{P}, \delta_2\mathbf{v}\cdot\mathbf{N}\big] = \frac{1}{c}\delta_1\boldsymbol{\varepsilon}\cdot\delta_2\mathbf{v}P^0, \tag{11b}$$

$$\frac{1}{i\hbar}\big[\delta_1\boldsymbol{\omega}\cdot\mathbf{J}, \delta_2\mathbf{v}\cdot\mathbf{N}\big] = (\delta_1\boldsymbol{\omega}\times\delta_2\mathbf{v})\cdot\mathbf{N}, \tag{11c}$$

$$\frac{1}{i\hbar}\big[\delta_1\mathbf{v}\cdot\mathbf{N}, \delta_2\mathbf{v}\cdot\mathbf{N}\big] = -\frac{1}{c^2}(\delta_1\mathbf{v}\times\delta_2\mathbf{v})\cdot\mathbf{J}. \tag{11d}$$

4. Compare the commutation relations in Einsteinian relativity with the Galilean ones. Thus, show that in Galilean relativity, $\mathbf{J}/c^2$ is neglected, and H is neglected relative to Mc^2, giving the effective replacement of the operator P^0/c by the number M.

References

[1] Julian Schwinger. *Particles, Sources and Fields: Vol. 1.* Addison-Wesley Publishing Company, Inc., 1970.