Homework No. 03 (Spring 2014)

PHYS 530A: Quantum Mechanics II

Due date: Wednesday, 2014 Feb 12, 4.30pm

1. A unitary matrix is defined by
$$U^\dagger U = 1, \tag{1}$$
where † stands for transpose and complex conjugation.
   (a) Show that
   $$U = e^{iH} \tag{2}$$
   is unitary if H is Hermitian, that is $H^\dagger = H$.
   (b) Show that
   $$U = \frac{1 + iA}{1 - iA} \tag{3}$$
   is unitary if A is Hermitian.
   (c) Using
   $$\tan^{-1} A = \frac{i}{2} \ln \left(\frac{1 - iA}{1 + iA} \right) \tag{4}$$
   show that
   $$H = 2 \tan^{-1} A. \tag{5}$$
   (d) Show that
   $$U = \frac{1 - iB}{1 + iB} \tag{6}$$
   is unitary if B is Hermitian.
2. Show that the combination $X^\dagger X$ is Hermitian, irrespective of X being Hermitian. Use this to deduce that the eigenvalues of $X^\dagger X$ are non-negative.
3. Prove that Hermitian operators have real eigenvalues. Further, show that any two eigenfunctions belonging to distinct (unequal) eigenvalues of a Hermitian operator are mutually orthogonal.
4. A is a matrix with eigenvalues $a_1, a_2, \ldots$, such that
$$\text{tr}A = \sum_i a_i, \tag{7}$$
exists. Prove that, for a function f,
$$\text{tr} f(A) = \sum_i f(a_i). \tag{8}$$

5. Let A and B be Hermitian operators.

(a) Let

$$C = \frac{1}{i}(AB - BA). \tag{9}$$

Show that C is Hermitian, that is $C^\dagger = C$.

(b) Let

$$A' = A - \overline{A}, \tag{10}$$
$$B' = B - \overline{B}, \tag{11}$$

where overline on an operator stands for mean of the operator, $\overline{A} = (\psi, A\psi)$. Show the following:

$$\overline{(A')^2} = (A'\psi, A'\psi) \geq 0, \tag{12}$$
$$\overline{(B')^2} = (B'\psi, B'\psi) \geq 0, \tag{13}$$
$$\overline{A'B'} = (A'\psi, B'\psi). \tag{14}$$

(c) Use the Schwarz inequality to learn

$$|(A'\psi, B'\psi)|^2 \leq (A'\psi, A'\psi)(B'\psi, B'\psi). \tag{15}$$

(d) Show that

$$(\psi, A'B'\psi) = \frac{1}{2}(\psi, (A'B' + B'A')\psi) + i\frac{1}{2}(\psi, C\psi). \tag{16}$$

(e) Show that $(\psi, (A'B' + B'A')\psi)$ and $(\psi, C\psi)$ are real. Thus, derive

$$|(A'\psi, B'\psi)|^2 = \frac{1}{4}|(\psi, (A'B' + B'A')\psi)|^2 + \frac{1}{4}|(\psi, C\psi)|^2. \tag{17}$$

(f) Using Eq. (17) in Eq. (15) derive the **uncertainty relation**

$$\frac{1}{2}\overline{C} \leq (\Delta A)(\Delta B), \tag{18}$$

where

$$(\Delta A)^2 = \overline{(A')^2} \quad \text{and} \quad (\Delta B)^2 = \overline{(B')^2} \tag{19}$$

are the mean square deviations (variance) in A and B.