Homework No. 04 (Spring 2014)

PHYS 530A: Quantum Mechanics II

Due date: Thursday, 2014 Feb 27, 4.30pm

1. (Ref: Milton's notes.) The energy of a charge e moving with velocity $\mathbf{v}$ in an external electromagnetic field is
$$E = e\phi - \frac{e}{c}\mathbf{v}\cdot\mathbf{A}, \tag{1}$$
where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. The relation between $\mathbf{A}$ and the magnetic field $\mathbf{H}$ is
$$\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{2}$$
For a constant (homogenous in space) magnetic field $\mathbf{H}$, verify that
$$\mathbf{A} = \frac{1}{2}\mathbf{H}\times\mathbf{r} \tag{3}$$
is a possible vector potential. Then, by looking at the energy, identify the magnetic moment $\boldsymbol{\mu}$ of the moving charge.

2. (Ref: Milton's notes.) Consider an atom entering a Stern-Gerlach apparatus. Deflection upward begins as soon as the atom enters the inhomogeneous field. By the time the atom leaves the field, it has been deflected upward by a net amount Δz. Compute Δz for
$$\mu_z = 10^{-27}\frac{\text{J}}{\text{G}}, \quad \frac{\partial H_z}{\partial z} = 10^6\frac{\text{G}}{\text{m}}, \quad l = 10\,\text{cm}, \quad T = \frac{mv_x^2}{k} = 10^3\,\text{K}. \tag{4}$$

3. (Ref: Milton's notes.) A silver atom has mass (actually the stable isotopes are Ag^{107}, Ag^{109})
$$m = 108\times 1.67\times 10^{-27}\,\text{kg}, \tag{5}$$
and speed
$$v = 10^2\,\text{m/s}. \tag{6}$$
Compute the reduced de Broglie wavelength, $\lambda\!\!\!^{-}$, and the corresponding diffraction angle $\delta\theta$ when a beam of such atoms passes through a slit of width $10^{-2}\,\text{cm}$. (See Fig. 3.3 in Milton's notes and discussion of Eq. (3.26) there.) Compare this diffraction angle with the deflection angle produced in a Stern-Gerlach experiment.

4. The Pauli matrices satisfy
$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \tag{7}$$
A particular representation of Pauli matrices is
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

(a) Construct the matrix

$$\sigma_{\theta,\phi} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} & -\cos\frac{\theta}{2} \end{pmatrix}, \tag{9}$$

where

$$\boldsymbol{\sigma} = \sigma_x\hat{\mathbf{i}} + \sigma_y\hat{\mathbf{j}} + \sigma_z\hat{\mathbf{k}}, \tag{10}$$

$$\hat{\mathbf{n}}(\theta,\phi) = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}. \tag{11}$$

Find the eigenvalues $\sigma'_{\theta,\phi}$ and the normalized eigenvectors, $|\sigma'_{\theta,\phi} = +1\rangle$ and $|\sigma'_{\theta,\phi} = -1\rangle$, (up to a phase) of the matrix $\sigma_{\theta,\phi}$.

(b) Now compute the matrices

$$\bar{\sigma}_x = \begin{pmatrix} \langle\sigma'_{\theta,\phi} = +1|\sigma_x|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = +1|\sigma_x|\sigma'_{\theta,\phi} = -1\rangle \\ \langle\sigma'_{\theta,\phi} = -1|\sigma_x|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = -1|\sigma_x|\sigma'_{\theta,\phi} = -1\rangle \end{pmatrix}, \tag{12}$$

$$\bar{\sigma}_y = \begin{pmatrix} \langle\sigma'_{\theta,\phi} = +1|\sigma_y|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = +1|\sigma_y|\sigma'_{\theta,\phi} = -1\rangle \\ \langle\sigma'_{\theta,\phi} = -1|\sigma_y|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = -1|\sigma_y|\sigma'_{\theta,\phi} = -1\rangle \end{pmatrix}, \tag{13}$$

$$\bar{\sigma}_z = \begin{pmatrix} \langle\sigma'_{\theta,\phi} = +1|\sigma_z|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = +1|\sigma_z|\sigma'_{\theta,\phi} = -1\rangle \\ \langle\sigma'_{\theta,\phi} = -1|\sigma_z|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = -1|\sigma_z|\sigma'_{\theta,\phi} = -1\rangle \end{pmatrix}. \tag{14}$$

These matrices are representation of the Pauli matrices in the eigenbasis of $\sigma_{\theta,\phi}$.

(c) Show that

$$\bar{\sigma}_i\bar{\sigma}_j = \delta_{ij} + i\varepsilon_{ijk}\bar{\sigma}_k. \tag{15}$$

5. (Ref: Milton's notes.) Using the notation for the probability for a measurement in the Stern-Gerlach experiment, introduced in the class, show that

$$p\big([+;\theta_1,\phi_1] \to [-;\theta_2,\phi_2]\big) = \frac{1-\cos\Theta}{2}, \tag{16}$$

where

$$\cos\Theta = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2). \tag{17}$$

6. Show that

$$p\big([+;0,0] \to [+;\pi,0]\big) = 0. \tag{18}$$

Further, show that

$$p\big([+;0,0] \to [\pm;\theta,\phi] \to [+;\pi,0]\big) = 0, \tag{19}$$

which is a statement of destructive interference. Compare this with the probability for

$$p\big([+;0,0] \to [+;\theta,\phi] \to [+;\pi,0]\big) \tag{20}$$

and

$$p\big([+;0,0] \to [-;\theta,\phi] \to [+;\pi,0]\big). \tag{21}$$

7. Show that

$$p\big([+;0,0] \to [-;\pi,0]\big) = 1. \tag{22}$$

Further, show that

$$p\big([+;0,0] \to [\pm;\theta,\phi] \to [-;\pi,0]\big) = 1, \tag{23}$$

which is a statement of constructive interference. Compare this with the probability for

$$p\big([+;0,0] \to [+;\theta,\phi] \to [-;\pi,0]\big) \tag{24}$$

and

$$p\big([+;0,0] \to [-;\theta,\phi] \to [-;\pi,0]\big). \tag{25}$$