Homework No. 05 (Spring 2014)

PHYS 530A: Quantum Mechanics II

Due date: Thursday, 2014 Mar 20, 4.30pm

1. (Based on Milton's lecture notes.) The vector product is defined by

$$(\mathbf{A} \times \mathbf{B})_3 = A_1 B_2 - A_2 B_1, \tag{1}$$

and similarly (cyclically) for the 1 and 2 components. If $A_{1,2,3}$ are elements of a non-commutative algebra, the components of $\mathbf{A}$ do not commute, and $\mathbf{A} \times \mathbf{A} \neq 0$ in general. Show that

$$\frac{\boldsymbol{\sigma}}{2} \times \frac{\boldsymbol{\sigma}}{2} = i \frac{\boldsymbol{\sigma}}{2}. \tag{2}$$

This is a special case of the angular momentum statement

$$\mathbf{J} \times \mathbf{J} = i\hbar \mathbf{J}. \tag{3}$$

2. (Based on Milton's lecture notes.) Consider a rotation of the coordinate system about the z-axis through an angle ϕ:

$$x' = x \cos\phi + y \sin\phi, \tag{4a}$$
$$y' = -x \sin\phi + y \cos\phi, \tag{4b}$$
$$z' = z. \tag{4c}$$

Pauli matrices, $\boldsymbol{\sigma}$, transform like a vector. The components of Pauli matrices transform as

$$\sigma_{x'} = \sigma_x \cos\phi + \sigma_y \sin\phi, \tag{5a}$$
$$\sigma_{y'} = -\sigma_x \sin\phi + \sigma_y \cos\phi, \tag{5b}$$
$$\sigma_{z'} = \sigma_z. \tag{5c}$$

Verify that the transformed components of Pauli matrices have the same algebraic properties as the original components:

$$\sigma_{x'}^2 = \sigma_{y'}^2 = 1, \qquad \sigma_{x'} \sigma_{y'} = i \sigma_{z'}. \tag{6}$$

3. The wavefunctions for the Stern-Gerlach experiment are

$$\psi_+(\theta, \phi) = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix} \quad \text{and} \quad \psi_-(\theta, \phi) = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}. \tag{7}$$

(a) Show that $\psi_-(\pi-\theta,\phi+\pi)=\psi_+(\theta,\phi)$, up to a phase.

(b) What is the physical interpretation?

4. The probability for a measurement in the Stern-Gerlach experiment is given by

$$p\big([+;\theta_1,\phi_1]\to[\pm;\theta_2,\phi_2]\big)=\frac{1\pm\cos\Theta}{2}, \tag{8}$$

where

$$\cos\Theta=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2). \tag{9}$$

Verify that

$$\mathrm{tr}\left(\frac{1+\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}(\theta_1,\phi_1)}{2}\right)\left(\frac{1\pm\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}(\theta_2,\phi_2)}{2}\right)=\frac{1\pm\cos\Theta}{2}. \tag{10}$$

5. Show that, for integer $n>0$,

$$z^n-1=(z-1)(1+z+z^2+\ldots+z^{n-1}). \tag{11}$$

If z's are the n-th roots of unity,

$$z^n=1, \tag{12}$$

with u'_k being the roots,

$$u'_k=e^{i\frac{2\pi}{n}k},\quad k=1,2,\ldots,n, \tag{13}$$

show that

$$z^n-1=\left(\frac{z}{u'_k}\right)^n-1=\left(\frac{z}{u'_k}-1\right)\left[1+\left(\frac{z}{u'_k}\right)+\left(\frac{z}{u'_k}\right)^2+\ldots+\left(\frac{z}{u'_k}\right)^{n-1}\right]. \tag{14}$$

6. In terms of the eigenvectors of the complementary variables, U and V, introduced in class, evaluate

$$\sum_{k=1}^{n}\langle v'_l|u'_k\rangle\langle u'_k|v'_m\rangle. \tag{15}$$

Thus, derive

$$\sum_{k=1}^{n}e^{i\frac{2\pi}{n}k(m-l)}=n\delta_{lm}. \tag{16}$$

(a) What do you learn about the n-th roots of unity from this relation?

(b) For, n=7, verify the (seven) relations after setting $l=n$.

(c) (Optional. Will not be graded.) Muse on the connection of the above relation with Ramanujan's sum.

(d) (Optional. Will not be graded.) Muse on the connection of the above relation with the discussion on 'Constructible polygons' in Wikipedia.