Homework No. 06 (Spring 2014)

PHYS 530A: Quantum Mechanics II

Due date: Not applicable

1. (Ref. Milton's notes.)
   (a) Consider three numerical vectors, **a**, **b**, **c**. Show that
   $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0. \tag{1}$$
   (b) Now consider operators A, B, C. Show that
   $$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0. \tag{2}$$
   (c) The multiplication property of the Pauli spin matrices can be written as
   $$(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b})=\mathbf{a}\cdot\mathbf{b}+i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}). \tag{3}$$
   From this, show that
   $$\frac{1}{i\hbar}\left[\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\mathbf{a},\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\mathbf{b}\right]=\frac{\hbar}{2}\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}). \tag{4}$$
   (d) More generally, what is
   $$\frac{1}{i\hbar}\left[\mathbf{J}\cdot\mathbf{a},\mathbf{J}\cdot\mathbf{b}\right]? \tag{5}$$
   (e) Use
   $$A=\frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{a},\qquad B=\frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{b},\qquad\text{and}\qquad C=\frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{c} \tag{6}$$
   in the result of problem (1b) to derive the result of problem (1a).

2. (Ref. Milton's notes.) A vector operator **V** is defined by the transformation property
   $$\frac{1}{i\hbar}\left[\mathbf{V},\delta\boldsymbol{\omega}\cdot\mathbf{J}\right]=\delta\boldsymbol{\omega}\times\mathbf{V}, \tag{7}$$
   which states the commutation relations of components of **V** with those of angular momentum **J**. Since a scalar operator S does not change under rotations it is defined by the corresponding transformation
   $$\frac{1}{i\hbar}\left[S,\delta\boldsymbol{\omega}\cdot\mathbf{J}\right]=0. \tag{8}$$
   (a) Show that the scalar product of vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a scalar.
   (b) Show that the vector product of vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a vector.