Homework No. 07 (Spring 2014)

PHYS 530A: Quantum Mechanics II

Due date: Thursday, 2014 Apr 17, 4.30pm

1. For $j = 1$:

   (a) Determine the matrix representantion for
   $$J_z, J_x, J_y, J_+, J_-, \text{and } J^2. \tag{1}$$

   (b) Evaluate
   $$\text{Tr}(J_k), \quad \text{Tr}(J_k J_l), \quad \text{and} \quad \text{Tr}(J_k^2 J_l^2), \quad \text{for} \quad k, l = x, y, z. \tag{2}$$

2. A quantum harmonic oscillator can be constructed out of two non-Hermitian operators, y and $y^\dagger$, that satisfy the commutation relation
   $$[y, y^\dagger] = 1. \tag{3}$$
   The eigenstate spectrum of the (Hermitian) number operator, $N = y^\dagger y$, represented by $|n\rangle$, where $n = 0, 1, 2, \ldots$, satisfy
   $$N|n\rangle = n|n\rangle, \qquad y|n\rangle = \sqrt{n}|n-1\rangle, \qquad y^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{4}$$

   (a) Build the matrix representation of the lowering operator using
   $$\langle n|y|n'\rangle = \begin{bmatrix} \langle 0|y|0\rangle & \langle 0|y|1\rangle & \langle 0|y|2\rangle & \langle 0|y|3\rangle & \langle 0|y|4\rangle & \cdots \\ \langle 1|y|0\rangle & \langle 1|y|1\rangle & \langle 1|y|2\rangle & \langle 1|y|3\rangle & \langle 1|y|4\rangle & \cdots \\ \langle 2|y|0\rangle & \langle 2|y|1\rangle & \langle 2|y|2\rangle & \langle 2|y|3\rangle & \langle 2|y|4\rangle & \cdots \\ \langle 3|y|0\rangle & \langle 3|y|1\rangle & \langle 3|y|2\rangle & \langle 3|y|3\rangle & \langle 3|y|4\rangle & \cdots \\ \langle 4|y|0\rangle & \langle 4|y|1\rangle & \langle 4|y|2\rangle & \langle 4|y|3\rangle & \langle 4|y|4\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5}$$
   Kindly calculate the first 5×5 block of the infinite dimensional matrix to report the pattern in the following questions.

   (b) Similarly, build the matrix representation of the raising operator $y^\dagger$.

   (c) Build the matrix representation of the number operator N.

   (d) Using the constructions
   $$y = \frac{1}{\sqrt{2\hbar}}(x + ip) \quad \text{and} \quad y^\dagger = \frac{1}{\sqrt{2\hbar}}(x - ip), \tag{6}$$
   determine the matrix representations for the Hermitian operators, x and p. Check that x and p are Hermitian matrices.

(e) Determine the matrices for the operators xp and px, and verify the commutation relation

$$\frac{1}{i\hbar}[x, p] = 1. \tag{7}$$

3. Using the asymptotic form for Hermite polynomials, $H_n(x)$, for large n, discuss the manner in which the harmonic oscillator eigenfunctions approach those of the free particle in the limit when the frequency of oscillations $\omega \to 0$.

4. (Set $\hbar = 1$.) From

$$y|n\rangle = \sqrt{n}|n-1\rangle \tag{8}$$

derive

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x). \tag{9}$$

Check this for $n = 4, 3, 2, 1, 0$. From

$$y^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{10}$$

derive

$$\left(2x - \frac{d}{dx}\right) H_n(x) = H_{n+1}(x). \tag{11}$$

Add the two statements to obtain

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x). \tag{12}$$

This recursion relation gives a way of recursively calculating $H_{n+1}(x)$ in terms of $H_n(x)$ and $H_{n-1}(x)$. Check this for $n = 3, 2, 1, 0$.

5. Use the results of Problem 4 to deduce the differential equation

$$\left(\frac{d^2}{dx^2} - 2x\frac{d}{dx} + 2n\right) H_n(x) = 0. \tag{13}$$

Show the equivalence of this with

$$\left(\frac{d^2}{dx^2} - x^2 + 2n + 1\right) \psi_n(x) = 0. \tag{14}$$

This is the "time-independent Schrödinger equation" for the harmonic oscillator.