SET-03

Infinitesimal canonical transformations

1. Hamilton's principle :

$$\delta \int_{t_1}^{t_2} dt \left[p\dot{q} - H \right] = 0 \quad \Rightarrow \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

2. Canonical transformations are transformations that preserve the Hamilton's principle.

$$K(\mathbb{P}, Q, t) = H(q, p, t)$$

$$Q = Q(q, p, t)$$
$$\mathbb{P} = \mathbb{P}(q, p, t)$$

such that

$$\delta \int_{t_1}^{t_2} dt \left[\mathbb{P}\dot{Q} - K \right] = 0 .$$

The requirement is the existance of a generating function,

$$W = W(q, Q, t)$$

such that

$$dW = p\,dq - \mathbb{P}\,dQ + (K - H)\,dt .$$

3. Poisson bracket,

$$[F, G] = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q},$$

is invariant under canonical transformations.

$$[F, G]_{p,q} = [F, G]_{\mathbb{P},Q} .$$

4 Equations of motion :

$$\boxed{\frac{dq}{dt} = \frac{\partial H}{\partial p} = [q, H]}$$
$$\boxed{\frac{dp}{dt} = -\frac{\partial H}{\partial q} = [p, H]}$$

Schrodinger equation
(differential equations)

Heisenberg equations
(algebraic)

5 Infinitesimal canonical transformations

$$Q = Q(q,p,t) = q + \delta q \qquad \delta q = \delta q(q,p,t)$$
$$\mathbb{P} = \mathbb{P}(q,p,t) = p + \delta p \qquad \delta p = \delta p(q,p,t)$$

6 The condition for the above infinitesimal transformation to be canonical is

$$\begin{aligned} dW &= p\, dq - \mathbb{P}\, dQ + (K-H)\, dt \\ &= p\, dq - (p+\delta p)\, d(q+\delta q) + (K-H)\, dt \\ &= - p\, d\delta q - \delta p\, dq + (K-H)\, dt \qquad + O(\delta)^2 . \end{aligned}$$

7 Let us define.

$$W(q, Q, t) = - p\, \delta q + \delta S$$

such that

$$dW = - p\, d\delta q - dp\, \delta q + d\delta S$$

8 Comparing 6 and 7 we have

$$d\,\delta S = -\delta p\, dq + \delta q\, dp + (K-H)\, dt$$

Thus, we conclude that

$$\delta S = \delta S(q, p, t)$$

and comparing with

$$d\,\delta S = \frac{\partial\, \delta S}{\partial q} dq + \frac{\partial\, \delta S}{\partial p} dp + \frac{\partial\, \delta S}{\partial t} \delta t$$

we have.

(i) $\delta q = \frac{\partial}{\partial p} \delta S$

(ii) $\delta p = -\frac{\partial}{\partial q} \delta S$

(iii) $K - H = \frac{\partial}{\partial t} \delta S$.

(9) Example:

Let $\delta S = H \underbrace{\delta t}_{\text{constant}}$

Then

$$\delta q = \frac{\partial}{\partial p}\delta S \Rightarrow \delta q = \frac{\partial}{\partial p} H \delta t = \delta t \frac{\partial H}{\partial p} = \delta t \frac{dq}{dt}$$

$$\delta p = -\frac{\partial}{\partial q}\delta S \Rightarrow \delta p = -\frac{\partial}{\partial q} H \delta t = -\delta t \frac{\partial H}{\partial q} = \delta t \frac{dp}{dt}$$

Thus, the interpretation is that motion is generated by successive infinitesimal transformations.

(10) <u>Example</u>: Translation in space

Let
$$\delta \vec{q} = \delta \vec{e}$$
$$\delta \vec{p} = 0$$

Then

$$\delta \vec{q} = \vec{\nabla}_p \delta S \Rightarrow \delta \vec{e} = \vec{\nabla}_p \delta S \Rightarrow \delta S = \vec{p}\cdot\delta\vec{e} + \text{const w.r.t } \vec{p}$$

$$\delta \vec{p} = -\vec{\nabla}_x \delta S \Rightarrow 0 = -\vec{\nabla}_x \delta S \Rightarrow \delta S = \text{const w.r.t } \vec{x}.$$

Thus, $\delta S = \vec{p}\cdot\delta\vec{e}$ $\qquad \vec{p}$ – linear momentum

(11) <u>Example</u> : Rotations

Let $\delta \vec{r} = \delta \vec{\omega} \times \vec{r}$

$\delta \vec{p} = \delta \vec{\omega} \times \vec{p}$

Then $\delta \vec{r} = \vec{\nabla}_p \, \delta S \Rightarrow \delta \vec{\omega} \times \vec{r} = \vec{\nabla}_p \, \delta S$

$\delta \vec{p} = - \vec{\nabla}_r \, \delta S \Rightarrow \delta \vec{\omega} \times \vec{p} = - \vec{\nabla}_r \, \delta S$

Consider $\delta S = \delta \vec{\omega} \cdot \vec{L} = \delta \vec{\omega} \cdot (\vec{r} \times \vec{p})$ $\quad \vec{L}$ - angular momentum

which verifies

$$\vec{\nabla}_p \, \delta S = \vec{\nabla}_p \, \delta \vec{\omega} \cdot (\vec{r} \times \vec{p}) = \delta \vec{\omega} \times \vec{r} \quad \checkmark$$

$$- \vec{\nabla}_r \, \delta S = - \vec{\nabla}_r \, \delta \vec{\omega} \cdot (\vec{r} \times \vec{p}) = \delta \vec{\omega} \times \vec{p} . \quad \checkmark$$

(12) How does an arbitrary function transform under infinitesimal canonical transformation?

$q = q(Q, \mathbb{P}, t)$
$p = p(Q, \mathbb{P}, t)$

$$
\begin{aligned}
F(q, p, t) &= F'(Q, \mathbb{P}, t) \\
&= F'(q + \delta q, p + \delta p, t) \\
&= F'(q, p, t) + \frac{\partial F'}{\partial q} \delta q + \frac{\partial F'}{\partial p} \delta p \qquad + O(\delta)^2 \\
&= F'(q, p, t) + \frac{\partial F'}{\partial q} \frac{\partial \, \delta S}{\partial p} - \frac{\partial F'}{\partial p} \frac{\partial \, \delta S}{\partial q} \\
&= F'(q, p, t) + [F', \delta S] \\
&= F'(q, p, t) + [F, \delta S] \qquad + O(\delta)^2 .
\end{aligned}
$$

(13) Example: Let $F = H$.

Then we have

$$H(q,p,t) = H'(q,p,t) + [H, \delta S]$$

$$= H'(q,p,t) - \frac{d\,\delta S}{dt} + \frac{\partial}{\partial t}\delta S$$

Thus, we read off that if

(i) H is form invariant, $H(q,p,t) = H'(q,p,t)$.

(ii) $\frac{\partial}{\partial t}\delta S = 0$, δS has no explicit time dependence.

then δS is a constant of motion,

$$\frac{d}{dt}\delta S = 0 .$$

This is the connection between form invariance (symmetry) with constants of motion (conservation laws).

(14) Example:

Consider $H(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(\vec{x}_1 - \vec{x}_2)$.

For $\delta\vec{x} = \delta\vec{e}$ = constant generated by $\delta S = (\vec{p}_1 + \vec{p}_2)\cdot\delta\vec{e}$

$\delta\vec{p} = 0$

we have

(i) H is form invariant under translation

(ii) δS is time independent

$$\Rightarrow \frac{d}{dt}\delta S = \frac{d}{dt}(\vec{p}_1+\vec{p}_2)\cdot\delta\vec{e} = \delta\vec{e}\cdot\frac{d}{dt}(\vec{p}_1+\vec{p}_2) = 0$$

$$\Rightarrow \frac{d}{dt}(\vec{p}_1+\vec{p}_2) = 0$$

statement of conservation of linear momentum.

(15) Example : Vector

Let $F = \vec{r}$ and $\delta S = \delta\vec{\omega} \cdot \vec{L} = \delta\vec{\omega} \cdot \vec{r} \times \vec{p}$.

This should tell us how the position vector transforms under rotations.

$$\vec{r} = \vec{r}\,' + [\vec{r}, \delta S]$$
$$= \vec{r}\,' + [\vec{r}, \delta\vec{\omega} \cdot \vec{r} \times \vec{p}]$$
$$= \vec{r}\,' + \delta\vec{\omega} \times \vec{r} .$$

$$\vec{A} = \vec{A}\,' + \delta\vec{\omega} \times \vec{A}$$

Any thing that transforms like position vector is defined as a vector.

(16) Example : Scalar

Let $F = \vec{x} \cdot \vec{x}$ and $\delta S = \delta\vec{\omega} \cdot \vec{L}$.

Then
$$\vec{x} \cdot \vec{x} = \vec{x}\,' \cdot \vec{x}\,' + [\vec{x} \cdot \vec{x}, \delta\vec{\omega} \cdot \vec{x} \times \vec{p}]$$
$$= \vec{x}\,' \cdot \vec{x}\,' + \vec{x} \cdot [\vec{x}, \delta\vec{\omega} \cdot \vec{L}] + [\vec{x}, \delta\vec{\omega} \cdot \vec{L}] \cdot \vec{x}$$
$$= \vec{x}\,' \cdot \vec{x}\,' + \vec{x} \cdot \delta\vec{\omega} \times \vec{x} + (\delta\vec{\omega} \times \vec{x}) \cdot \vec{x}$$
$$= \vec{x}\,' \cdot \vec{x}\,' + \underbrace{\delta\vec{\omega} \cdot (\vec{x} \times \vec{x})}_{=0} + \underbrace{\delta\vec{\omega} \cdot (\vec{x} \times \vec{x})}_{=0}$$
$$= \vec{x}\,' \cdot \vec{x}\,'$$

Also. $\vec{A} \cdot \vec{B} = \vec{A}\,' \cdot \vec{B}\,'$ $\longrightarrow$ show this as a home work.

Definition of scalar :
$$F(\vec{q}, \vec{p}, t) = F'(\vec{q}, \vec{p}, t)$$ — form invariance under rotation

$$\Rightarrow \quad [F, \delta\vec{\omega} \cdot \vec{L}] = 0$$

(17) Example:

Consider $H(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(|\vec{x}_1 - \vec{x}_2|)$

$\delta \vec{x}_1 = \delta \vec{\omega} \times \vec{x}_1$ $\qquad$ $\delta \vec{x}_2 = \delta \vec{\omega} \times \vec{x}_2$

$\delta \vec{p}_1 = \delta \vec{\omega} \times \vec{p}_1$ $\qquad$ $\delta \vec{p}_2 = \delta \vec{\omega} \times \vec{p}_2$

$$\delta S = \delta \vec{\omega} \cdot \vec{L}$$

Thus, (i) H is form invariant under rotations

(ii) δS has not explicit time

$$\Rightarrow \quad \frac{d}{dt}\delta S = 0$$

$$\frac{d}{dt}(\delta \vec{\omega} \cdot \vec{L}) = 0$$

$$\delta \vec{\omega} \cdot \frac{d\vec{L}}{dt} = 0 \qquad \delta \vec{\omega} = \text{constant}.$$

$$\frac{d\vec{L}}{dt} = 0$$

$\Rightarrow$ conservation of angular momentum.

(18) Let us consider the Poisson bracket under infinitesimal canonical transformations

$$[F, G]_{qp} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}$$

$$\begin{aligned}
\frac{\partial F}{\partial q} &= \frac{\partial F}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial \mathbb{P}}{\partial q} \\
&= \frac{\partial F}{\partial Q}\frac{\partial}{\partial q}\left(q + \delta q\right) + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial}{\partial q}(p + \delta p) \\
&= \frac{\partial F}{\partial Q} + \frac{\partial F}{\partial Q}\frac{\partial}{\partial q}\frac{\partial}{\partial p}\delta S - \frac{\partial F}{\partial \mathbb{P}}\frac{\partial}{\partial q}\frac{\partial}{\partial p}\delta S.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F}{\partial p} &= \frac{\partial F}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial \mathbb{P}}{\partial p} \\
&= \frac{\partial F}{\partial Q}\frac{\partial}{\partial p}(q + \delta q) + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial}{\partial p}(p + \delta p) \\
&= \frac{\partial F}{\partial \mathbb{P}} + \frac{\partial F}{\partial Q}\frac{\partial}{\partial p}\frac{\partial}{\partial p}\delta S - \frac{\partial F}{\partial \mathbb{P}}\frac{\partial}{\partial p}\frac{\partial}{\partial q}\delta S.
\end{aligned}$$

$$\frac{\partial G}{\partial q} = \frac{\partial G}{\partial Q} + \frac{\partial G}{\partial Q}\frac{\partial}{\partial q}\frac{\partial}{\partial p}\delta S - \frac{\partial G}{\partial \mathbb{P}}\frac{\partial}{\partial q}\frac{\partial}{\partial p}\delta S.$$

$$\frac{\partial G}{\partial p} = \frac{\partial G}{\partial \mathbb{P}} + \frac{\partial G}{\partial Q}\frac{\partial}{\partial p}\frac{\partial}{\partial p}\delta S - \frac{\partial G}{\partial \mathbb{P}}\frac{\partial}{\partial p}\frac{\partial}{\partial q}\delta S.$$

(19)
$$\begin{aligned}
[F, G]_{q,p} = {} & \frac{\partial F}{\partial Q}\frac{\partial G}{\partial \mathbb{P}} + \frac{\partial F}{\partial Q}\frac{\partial G}{\partial Q}\frac{\partial^2}{\partial p^2}\delta S - \frac{\partial F}{\partial Q}\frac{\partial G}{\partial \mathbb{P}}\frac{\partial}{\partial p}\frac{\partial}{\partial q}\delta S. \\
& + \frac{\partial F}{\partial Q}\frac{\partial G}{\partial \mathbb{P}}\frac{\partial}{\partial q}\frac{\partial}{\partial p}\delta S - \frac{\partial F}{\partial \mathbb{P}}\frac{\partial G}{\partial \mathbb{P}}\frac{\partial}{\partial q}\frac{\partial}{\partial p}\delta S \\
& - \frac{\partial F}{\partial \mathbb{P}}\frac{\partial G}{\partial Q} - \frac{\partial F}{\partial \mathbb{P}}\frac{\partial G}{\partial Q}\frac{\partial}{\partial q}\frac{\partial}{\partial p}\delta S. + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial G}{\partial \mathbb{P}}\frac{\partial}{\partial q}\frac{\partial}{\partial p}\delta S. \\
& - \frac{\partial F}{\partial Q}\frac{\partial G}{\partial Q}\frac{\partial}{\partial p}\frac{\partial}{\partial p}\delta S + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial G}{\partial Q}\frac{\partial}{\partial p}\frac{\partial}{\partial q}\delta S. \\
= {} & [F, G]_{Q,\mathbb{P}} \qquad \text{Q.E.D.}
\end{aligned}$$